A CATEGORIFICATION OF $\mathbf{U}_T(\mathfrak{sl}(1|1))$ AND ITS TENSOR PRODUCT REPRESENTATIONS

YIN TIAN

ABSTRACT. We describe the super Hopf algebra $\mathbf{U}_T(\mathfrak{sl}(1|1))$, which is a variant of the super quantum group $\mathbf{U}_q(\mathfrak{sl}(1|1))$, and its representations V_n for $n > 0$. We construct families of differential graded algebras A, B and R_n, and consider the DG categories $DGP(A)$, $DGP(B)$ and $DGP(R_n)$, which are full DG subcategories of the categories of DG A-, B- and R_n-modules generated by certain distinguished projective modules. The 0-th homology categories $H^0(DGP(A))$, $H^0(DGP(B))$, and $H^0(DGP(R_n))$ are triangulated and can be viewed as algebraic reformulations of the contact categories of an annulus, a twice punctured disk, and an n times punctured disk. Their Grothendieck groups are isomorphic to $\mathbf{U}_T(\mathfrak{sl}(1|1))$, $\mathbf{U}_T(\mathfrak{sl}(1|1)) \otimes_{\mathbb{Z}} \mathbf{U}_T(\mathfrak{sl}(1|1))$ and V_n, respectively. We categorify the multiplication and comultiplication on $\mathbf{U}_T(\mathfrak{sl}(1|1))$ to a bifunctor $H^0(DGP(A)) \times H^0(DGP(A)) \to H^0(DGP(A))$ and a functor $H^0(DGP(A)) \to H^0(DGP(B))$, respectively. We also categorify the $\mathbf{U}_T(\mathfrak{sl}(1|1))$-action on V_n to a bifunctor $H^0(DGP(A)) \times H^0(DGP(R_n)) \to H^0(DGP(R_n))$.

1. INTRODUCTION

1.1. **Background.** This paper is a sequel to [36] in which we categorified the algebra structure of the super quantum group $\mathbf{U}_q(\mathfrak{sl}(1|1))$. The goal of this paper is to present a categorification of a super Hopf algebra $\mathbf{U}_T(\mathfrak{sl}(1|1))$ (a variant of $\mathbf{U}_q(\mathfrak{sl}(1|1))$) and its representations $V_n = V_1^{\otimes n}$ for $n > 0$, where V_1 is the two-dimensional fundamental representation.

In the late 1980's, Witten [41] and Reshetikhin-Turaev [32] established a connection between quantum groups and knot invariants. In particular, the Jones polynomial could be recovered as the Witten-Reshetikhin-Turaev invariant of the fundamental representation of $\mathbf{U}_q(\mathfrak{sl}_2)$. For super quantum groups, Kauffman and Saleur in [15] developed an analogous representation-theoretic approach to the Alexander polynomial, by considering the fundamental representation V_1 of $\mathbf{U}_q(\mathfrak{sl}(1|1))$. Rozansky and Saleur in [31] gave a corresponding quantum field theory description.

The connection between quantum groups and knot invariants can be lifted to the categorical level. The existence of such a lifting process, called *categorification*, was conjectured by Crane and Frenkel in [4]. In the seminal paper [17], Khovanov defined a doubly graded homology, now

called *Khovanov homology*, whose graded Euler characteristic agreed with the Jones polynomial. A different categorification of the Jones polynomial, called *odd Khovanov homology*, was later discovered by Ozsváth-Rasmussen-Szabó [27]. Chuang and Rouquier [5] categorified locally finite $\mathfrak{sl}_2$-representations, and more generally, Rouquier [33] constructed a 2-category associated with a Kac-Moody algebra. For the quantum groups themselves, Lauda [22] gave a diagrammatic categorification of $\mathbf{U}_q(\mathfrak{sl}_2)$ and Khovanov-Lauda [19, 20, 21] extended it to the cases of $\mathbf{U}_q(\mathfrak{sl}_n)$ and one-half of the quantum groups associated to an arbitrary Cartan datum. The program of categorifying Witten-Reshetikhin-Turaev invariants was brought to fruition by Webster [38, 39] using the diagrammatic approach.

On the other hand, a categorification of the Alexander polynomial, called *knot Floer homology*, was defined independently by Ozsváth-Szabó [28] and Rasmussen [30]. Although its initial definition was through Lagrangian Floer homology, knot Floer homology was shown to admit a completely combinatorial description by Manolescu-Ozsváth-Sarkar [26]. It is therefore natural to ask whether there is a categorical program for $\mathbf{U}_q(\mathfrak{sl}(1|1))$ which is analogous to that of the $\mathbf{U}_q(\mathfrak{sl}_2)$ case and which recovers knot Floer homology.

This paper presents another step towards such a categorical program, namely the categorification of the multiplication and comultiplication on $\mathbf{U}_T(\mathfrak{sl}(1|1))$ and the representations V_n of $\mathbf{U}_T(\mathfrak{sl}(1|1))$. In a subsequent paper [37], we categorify the action of the braid group on V_n which is induced by the R-matrix structure of $\mathbf{U}_T(\mathfrak{sl}(1|1))$. It should be mentioned that Sartori [34] has recently announced a categorification of tensor products of the fundamental representation of $\mathfrak{gl}(1|1)$ using completely different methods.

1.2. **Main results.** The super Hopf algebra $\mathbf{U}_T(\mathfrak{sl}(1|1))$ is a variant of $\mathbf{U}_q(\mathfrak{sl}(1|1))$ which is an associative $\mathbb{Z}$-algebra with unit I, generators E, F, T, T^{-1} and relations:

$$E^2 = F^2 = 0, \tag{1}$$

$$EF + FE = I - T, \tag{2}$$

$$ET = TE,\ FT = TF, \tag{3}$$

$$TT^{-1} = T^{-1}T = I. \tag{4}$$

The comultiplication $\Delta : \mathbf{U}_T(\mathfrak{sl}(1|1)) \to \mathbf{U}_T(\mathfrak{sl}(1|1)) \otimes_{\mathbb{Z}} \mathbf{U}_T(\mathfrak{sl}(1|1))$ is given by:

$$\Delta(E) = E \otimes I + I \otimes E,$$

$$\Delta(F) = F \otimes T + I \otimes F,$$

$$\Delta(T) = T \otimes T.$$

Recall from [15] that the commutator relation of $\mathbf{U}_q(\mathfrak{sl}(1|1))$ is:

$$EF + FE = \frac{H - H^{-1}}{q - q^{-1}}. \tag{5}$$

To see the relation between $\mathbf{U}_T(\mathfrak{sl}(1|1))$ and $\mathbf{U}_q(\mathfrak{sl}(1|1))$, we compare their commutator relations, Equations (2) and (5), by setting $T = H^{-2}$. Then Equation (5) is equal to the multiplication of Equation (2) by $\frac{H}{q-q^{-1}}$.

Let $\mathbf{U}_T$ denote $\mathbf{U}_T(\mathfrak{sl}(1|1))$ from now on. Consider the fundamental representation V_1 of $\mathbf{U}_T$, a free $\mathbb{Z}[t^{\pm 1}]$-module generated by $\mathcal{B}_1 = \{|0\rangle, |1\rangle\}$ satisfying:

$$E|0\rangle = 0, \quad F|0\rangle = |1\rangle,$$
$$E|1\rangle = (1-t)|0\rangle, \quad F|1\rangle = 0,$$
$$T|0\rangle = t|0\rangle, \quad T|1\rangle = t|1\rangle.$$

Similarly, consider its n-th tensor product representation $V_n = V_1^{\otimes n}$ induced by the iterated comultiplication of $\mathbf{U}_T$. There is a distinguished basis $\mathcal{B}'_n$ of V_n:

$$\mathcal{B}'_n = \mathcal{B}_1^{\times n} = \{\mathbf{a} = |a_1 \dots a_n\rangle \mid a_i \in \{0,1\}\},$$

where $|a_1 \dots a_n\rangle$ is the shorthand for $|a_1\rangle \otimes \dots \otimes |a_n\rangle$. Note that $T \cdot v = t^n v$ for $v \in V_n$ since $\Delta(T) = T \otimes T$.

The following are the main results of this paper:

Theorem 1.1 (Categorification of the multiplication on $\mathbf{U}_T$)**.** *There exist a triangulated category $H^0(DGP(A))$ whose Grothendieck group is $\mathbf{U}_T$ and an exact bifunctor*

$$\mathcal{M} : H^0(DGP(A)) \times H^0(DGP(A)) \to H^0(DGP(A))$$

whose induced map $K_0(\mathcal{M}) : \mathbf{U}_T \times \mathbf{U}_T \to \mathbf{U}_T$ on the Grothendieck groups agrees with the multiplication on $\mathbf{U}_T$.

Theorem 1.2 (Categorification of the comultiplication on $\mathbf{U}_T$)**.** *There exist a triangulated category $H^0(DGP(B))$ whose Grothendieck group is $\mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$ and an exact functor*

$$\delta : H^0(DGP(A)) \to H^0(DGP(B))$$

whose induced map $K_0(\delta) : \mathbf{U}_T \to \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$ on the Grothendieck groups agrees with the comultiplication on $\mathbf{U}_T$.

Theorem 1.3 (Categorification of the $\mathbf{U}_T$-module V_n). *For each $n > 0$, there exist a triangulated category $H^0(DGP(R_n))$ whose Grothendieck group is V_n and an exact bifunctor*

$$\mathcal{M}_n : H^0(DGP(A)) \times H^0(DGP(R_n)) \to H^0(DGP(R_n))$$

whose induced map $K_0(\mathcal{M}_n)$ on the Grothendieck groups agrees with the action $\mathbf{U}_T \times V_n \to V_n$.

1.3. **Motivation.** The motivation is from the *contact category* introduced by Honda [8] which presents an algebraic way to study contact topology in dimension 3. The contact category $\mathcal{C}(\Sigma, F)$ of (Σ, F) is an additive category associated to a compact oriented surface Σ and a finite subset F of $\partial\Sigma$. The objects of $\mathcal{C}(\Sigma, F)$ are isotopy classes of *dividing sets* on Σ whose restrictions to $\partial\Sigma$ agree with F. The morphisms are generated by *tight* contact structures on $\Sigma \times [0, 1]$ with prescribed dividing sets on $\Sigma \times \{0, 1\}$. More precisely, a dividing set on Σ is a properly embedded 1-manifold, possibly disconnected and possibly with boundary, which divides Σ into positive and negative regions. Any dividing set with a contractible component is defined as the zero object since there is no tight contact structure in a neighborhood of the dividing curve by a criterion of Giroux [7]. As basic blocks of morphisms, *bypass attachments* introduced by Honda [9] locally change dividing sets as shown in Figure 1. Honda-Kazez-Matić [10] gave a criterion for the addition of a collection of disjoint bypasses to be tight.

A morphism set generated by tight contact structures in $\Sigma \times I$ is closely related to *sutured Floer homology* of $\Sigma \times I$ which is defined by Juhász [14]. The connection between 3-dimensional contact topology and *Heegaard Floer homology* was established by Ozsváth and Szabó [29] in the closed case. Honda-Kazez-Matić generalized it to the case of a contact 3-manifold with *convex* boundary in [12] and formulated it in the framework of TQFT in [11]. The combinatorial properties of this TQFT formulation were intensively studied by Mathews in the case of disks [24] and annuli [25]. The connection on the categorical level is observed by Zarev [43].

There is a refined version, called the *universal cover* $\tilde{\mathcal{C}}(\Sigma, F)$ of the contact category $\mathcal{C}(\Sigma, F)$. The objects of $\tilde{\mathcal{C}}(\Sigma, F)$ are isotopy classes of dividing sets on (Σ, F) equipped with a grading taking values in homotopy classes of 2-plane fields on $\Sigma \times [0, 1]$. Equivalently, the grading $\mathrm{gr}(\Sigma)$ is given by a central extension by $\mathbb{Z}$ of the homology group $H_1(\Sigma)$, i.e., there is a short exact sequence:

$$0 \to \mathbb{Z} \to \mathrm{gr}(\Sigma) \to H_1(\Sigma) \to 0.$$

Note that a similar grading appears in bordered Heegaard Floer homology [23, Section 3.3]. The morphisms of $\tilde{\mathcal{C}}(\Sigma, F)$ are generated by tight contact structures on $\Sigma \times [0, 1]$ which are compatible with the grading. The main feature of the universal cover $\tilde{\mathcal{C}}(\Sigma, F)$ is the existence of distinguished

triangles given by a triple of bypass attachments as shown in Figure 1. The subgroup $\mathbb{Z}$ of the grading $\mathrm{gr}(\Sigma)$ is related to the shift functor in a triangulated category. In particular, Huang [13] showed that a triple of bypass attachments changes that component by 1.

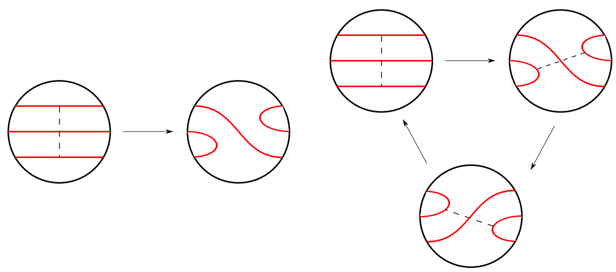

FIGURE 1. The left picture is a bypass attachment; the right one is a distinguished triangle given by a triple of bypass attachments.

This paper provides an algebraic reformulation of the universal covers of the contact categories of an annulus, a twice punctured disk and an n times punctured disk. Let $\tilde{\mathcal{C}}_o$ be the universal cover of the contact category $\mathcal{C}(S_o, F_o)$, where S_o is an annulus and F_o consists of two points on each boundary component. Then $\tilde{\mathcal{C}}_o$ is a monoidal category with a bifunctor $\mathcal{M} : \tilde{\mathcal{C}}_o \times \tilde{\mathcal{C}}_o \to \tilde{\mathcal{C}}_o$ defined by stacking two dividing sets along their common boundaries of two annuli for objects and gluing two contact structures for morphisms. See Figure 2. A distinguished basis of the Grothendieck group $K_0(\tilde{\mathcal{C}}_o)$ is given by the classes of dividing sets $\mathcal{B} = \{I, E, F, EF\}$, where EF is the stacking of E and F under the monoidal functor $\mathcal{M}$. The generator of $H_1(S_o)$ in the grading $\mathrm{gr}(S_o)$ corresponds to the central element $T \in \mathbf{U}_T$. The commutator relation in Equation (2) comes from two

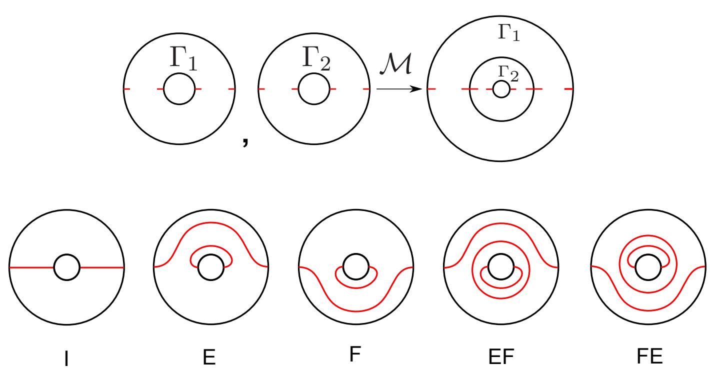

FIGURE 2. The upper picture describes the monoidal functor $\mathcal{M}$ on objects; the lower one consists of the distinguished basis of $K_0(\tilde{\mathcal{C}}_o)$ and the dividing set FE.

distinguished triangles connecting I, T, EF and FE in $\tilde{\mathcal{C}}_o$. Then $K_0(\tilde{\mathcal{C}}_o)$ is isomorphic to $\mathbf{U}_T$ and the monoidal functor $\mathcal{M}$ categorifies the multiplication on $\mathbf{U}_T$.

To categorify the comultiplication $\Delta : \mathbf{U}_T \to \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$, we look for some contact category whose Grothendieck group is isomorphic to $\mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$. Let $\tilde{\mathcal{C}}_{oo}$ be the universal cover of the contact category $\mathcal{C}(S_{oo}, F_{oo})$, where S_{oo} is a twice punctured disk and F_{oo} consists of two points on each boundary component. A distinguished basis of the Grothendieck group $K_0(\tilde{\mathcal{C}}_{oo})$ is given by the classes of dividing sets $\{\Gamma_1 \otimes \Gamma_2 \mid \Gamma_1, \Gamma_2 \in \mathcal{B}\}$ as shown in Figure 3. There are two generators $t_1, t_2 \in H_1(S_{oo})$ in the grading $\mathrm{gr}(S_{oo})$ given by the loops. They correspond to the central elements $T \otimes I, I \otimes T \in \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$. Hence the Grothendieck group $K_0(\tilde{\mathcal{C}}_{oo})$ is isomorphic to $\mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$. Define a functor $\delta : \tilde{\mathcal{C}}_o \to \tilde{\mathcal{C}}_{oo}$ on objects by stacking dividing sets $\Gamma \in \tilde{\mathcal{C}}_o$ with the specific dividing set $I \otimes I \in \tilde{\mathcal{C}}_{oo}$ along the outmost boundary of S_{oo}, on morphisms by gluing contact structures in $S_o \times [0,1]$ with the I*-invariant* contact structure of $I \otimes I$ in $S_{oo} \times [0,1]$. Then the decategorification $K_0(\delta)$ can be viewed as a map $\mathbf{U}_T \to \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$. It turns out that $K_0(\delta)$ agree with the comultiplication Δ. For instance, $\Delta(E) = E \otimes I + I \otimes E$ is given by a distinguished triangle connecting $\delta(E), E \otimes I$ and $I \otimes E$ in $\tilde{\mathcal{C}}_{oo}$.

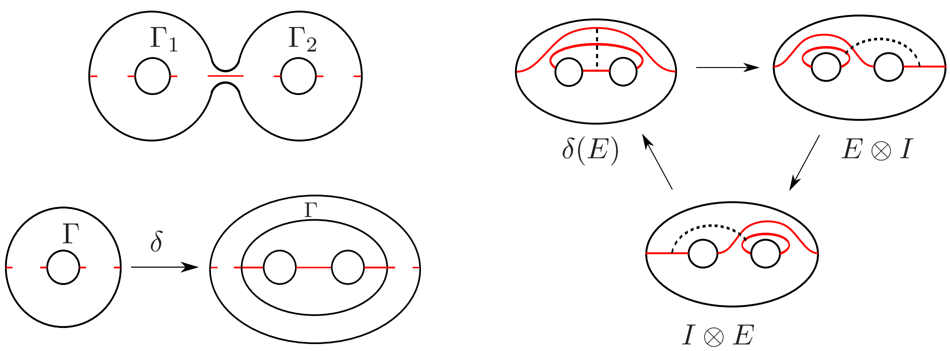

FIGURE 3. The upper picture on the left describes the basis $\{\Gamma_1 \otimes \Gamma_2\}$ of $K_0(\tilde{\mathcal{C}}_{oo})$; the lower picture on the left gives the comultiplication δ; the picture on the right is the triangle of $\delta(E), E \otimes I$ and $I \otimes E$ in $\tilde{\mathcal{C}}_{oo}$.

Let $\tilde{\mathcal{C}}_n$ be the universal cover of the contact category of (Σ_n, F_n), where Σ_n is an n times punctured disk and F_n contains two marked points on the outermost boundary and no points on the other boundary components[1]. The Grothendieck group $K_0(\tilde{\mathcal{C}}_n)$ is a free module over $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_n^{\pm 1}]$, where t_i is the generator in $H_1(\Sigma_n)$ corresponding to the i-th loop. A quotient of $K_0(\tilde{\mathcal{C}}_n)$ by the

[1]Since there is no marked point on interior boundary components $\partial\Sigma_n'$, the boundary restriction of contact structures in $\mathrm{Hom}_{\tilde{\mathcal{C}}_n}(\Gamma_1, \Gamma_2)$ is a collection of dividing sets $(\partial\Sigma_n' \times \{1/2\} \cup \Gamma_1 \times \{0\} \cup \Gamma_2 \times \{1\}) \subset \partial(\Sigma \times [0,1])$.

relation $t_1 = t_2 = \cdots = t_n = t$ is isomorphic to the $\mathbf{U}_T$-module V_n. A distinguished collection of dividing sets in $\tilde{\mathcal{C}}_n$ is obtained by lifting the basis $\mathcal{B}'_n$ of $V_n = V_1^{\otimes n}$. See Figure 4 for $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$. Note that $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_o$ have the same underlying surface but with different boundary conditions. The categorical $\mathbf{U}_T$-action on V_n is a functor $\mathcal{M}_n : \tilde{\mathcal{C}}_o \times \tilde{\mathcal{C}}_n \to \tilde{\mathcal{C}}_n$ given by stacking dividing sets on the annulus S_o with dividing sets on the n times punctured disk Σ_n along its outermost boundary. Consider morphism sets between the dividing sets in the distinguished collections. For instance, there is a unique tight contact structure $e_\emptyset \in \mathrm{Hom}_{\tilde{\mathcal{C}}_1}(|0\rangle, |0\rangle)$ which is an idempotent element. There are exactly two tight contact structures $e_1, \rho_1 \in \mathrm{Hom}_{\tilde{\mathcal{C}}_1}(|1\rangle, |1\rangle)$, where e_1 is an idempotent element and ρ_1 is a nilpotent element:

$$\mathrm{Hom}_{\tilde{\mathcal{C}}_1}(|0\rangle, |0\rangle) = \langle e_\emptyset \rangle; \quad \mathrm{Hom}_{\tilde{\mathcal{C}}_1}(|1\rangle, |1\rangle) = \langle e_1, \rho_1 | \rho_1^2 = 0 \rangle.$$

Similarly, nontrivial morphism sets in $\tilde{\mathcal{C}}_2$ are the following:

$$\begin{gathered}
\mathrm{Hom}_{\tilde{\mathcal{C}}_2}(|00\rangle, |00\rangle) = \langle e_\emptyset \rangle; \\
\mathrm{Hom}_{\tilde{\mathcal{C}}_2}(|10\rangle, |10\rangle) = \langle e_1, \rho_1 \mid \rho_1^2 = 0 \rangle, \\
\mathrm{Hom}_{\tilde{\mathcal{C}}_2}(|01\rangle, |01\rangle) = \langle e_2, \rho_2 \mid \rho_2^2 = 0 \rangle, \\
\mathrm{Hom}_{\tilde{\mathcal{C}}_2}(|01\rangle, |10\rangle) = \langle r,\ \rho_2 \cdot r,\ r \cdot \rho_1,\ \rho_2 \cdot r \cdot \rho_1 \mid \rho_1^2 = \rho_2^2 = 0 \rangle; \\
\mathrm{Hom}_{\tilde{\mathcal{C}}_2}(|11\rangle, |11\rangle) = \langle e_{1,2},\ \rho_1,\ \rho_2,\ \rho_1 \cdot \rho_2 \mid \rho_1^2 = \rho_2^2 = 0,\ \rho_1 \cdot \rho_2 = \rho_2 \cdot \rho_1 \rangle.
\end{gathered}$$

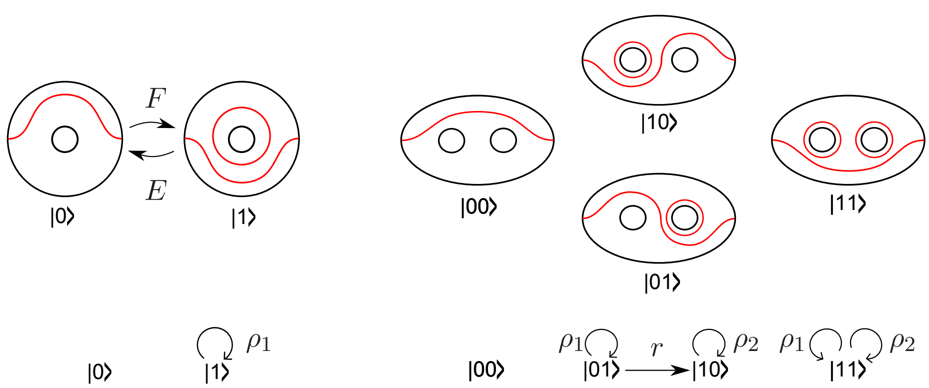

FIGURE 4. The upper picture gives the distinguished collections of dividing sets in $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$. The quivers in the lower picture describe morphisms in $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$.

At this point we pass to algebra[2]. An algebraic reformulation of the universal cover of the contact categories $\tilde{\mathcal{C}}_n$ is given as follows. We construct a quiver Γ_n whose vertex set $V(\Gamma_n)$ is the basis $\mathcal{B}'_n$ of V_n. The arrow set $A(\Gamma_n)$ is given by morphisms between the objects of $\tilde{\mathcal{C}}_n$ in the distinguished collection lifting the basis $\mathcal{B}'_n$. Consider the path algebra $\mathbb{F}_2\Gamma_n$ of the quiver Γ_n with an additional t-grading, where $\mathbb{F}_2$ is the field of two elements. Finally, we construct a t-graded DG algebra R_n by introducing a nontrivial differential to a quotient of $\mathbb{F}_2\Gamma_n$. We prove that R_n is formal, i.e., it is quasi-isomorphic to its cohomology $H(R_n)$. Similarly, we construct t-graded DG algebras A and B from the contact categories $\tilde{\mathcal{C}}_o$ and $\tilde{\mathcal{C}}_{oo}$.

The DG algebra R_n is closely related to the *strands algebra* in bordered Heegaard Floer homology [23] for an n times punctured disk. In general, the strands algebra of any surface with boundary was defined by Zarev [42]. Motivated by the *rook monoid* [35] and its diagrammatic presentation *rook diagrams* [6], we describe R_n in terms of *decorated rook diagrams*. The rook diagram is used to study the Alexander and Jones polynomials by Bigelow-Ramos-Yi [3], and tensor representations of $\mathfrak{gl}(1|1)$ by Benkart-Moon [1]. Motivated by the strands algebra, Khovanov [18] categorified the positive part of $\mathbf{U}_q(\mathfrak{gl}(1|2))$.

Consider a DG category $DG(R_n)$ of t-graded DG R_n-modules. There is a full subcategory $DGP(R_n)$ generated by some distinguished projective DG R_n-modules. We model $\tilde{\mathcal{C}}_n$ by the 0-th homology category $H^0(DGP(R_n))$ which is equivalent to $H^0(DGP(H(R_n)))$ as triangulated categories. Their Grothendieck groups are isomorphic to free $\mathbb{Z}[t^{\pm 1}]$-modules over the vertex set $\mathcal{B}'_n$:

$$K_0(H^0(DGP(R_n))) \cong K_0(H^0(DGP(H(R_n)))) \cong \mathbb{Z}[t^{\pm 1}]\langle \mathcal{B}'_n \rangle \cong V_n.$$

Similarly, $\tilde{\mathcal{C}}_o$ and $\tilde{\mathcal{C}}_{oo}$ are modeled by triangulated categories $H^0(DGP(A))$ and $H^0(DGP(B))$ whose Grothendieck groups are isomorphic to $\mathbf{U}_T$ and $\mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$, respectively.

In order to categorify the $\mathbf{U}_T$-action on V_n, we define a DG algebra $A \boxtimes R_n$ by adding a differential to $A \otimes R_n$. Consider a triangulated category $H^0(DGP(A \boxtimes R_n))$ whose Grothendieck group is isomorphic to a quotient $\mathbf{U}_T \otimes_{\{T=t^n\}} V_n$ of $\mathbf{U}_T \times V_n$ by the relation $(T, v) = (I, t^n v)$. Motivated by stacking of dividing sets in the contact categories, we define a DG $(H(R_n), A \boxtimes R_n)$-bimodule C_n which is the key construction in our categorification. A functor defined by tensoring with C_n over $A \boxtimes R_n$: $DGP(A \boxtimes R_n) \xrightarrow{C_n \otimes -} DGP(H(R_n))$ induces an exact functor η_n between their 0-th homology categories. The decategorification $K_0(\eta_n)$ on the Grothendieck groups agrees with the $\mathbf{U}_T$-action on V_n: $\mathbf{U}_T \otimes_{\{T=t^n\}} V_n \to V_n$. Similarly, we construct functors $\eta : H^0(DGP(A \otimes A)) \xrightarrow{T \otimes -} H^0(DGP(A))$ by tensoring with a DG $(A, A \otimes A)$-bimodule T, and

[2] In fact, the rest of this paper is just algebra which is motivated by the contact category.

$\delta : H^0(DGP(A)) \xrightarrow{S\otimes -} H^0(DGP(B))$ by tensoring with a DG (B, A)-bimodule S. We show that η and δ categorify the multiplication and comultiplication on $\mathbf{U}_T$, respectively.

$$\begin{array}{lccc} \eta_n : & H^0(DGP(A \boxtimes R_n)) & \xrightarrow{C_n\otimes -} & H^0(DGP(H(R_n))) \\ & \downarrow K_0 & & \downarrow K_0 \\ K_0(\eta_n) : & \mathbf{U}_T \otimes_{\{T=t^n\}} V_n & \longrightarrow & V_n. \end{array}$$

The organization of the paper. In Section 2 we define the super Hopf algebra $\mathbf{U}_T$ and categorify its multiplication via the DG $(A, A \otimes A)$-bimodule T. In Section 3 we categorify the comultiplication on $\mathbf{U}_T$ via the DG (B, A)-bimodule S. In Section 4 we define the fundamental representation V_1 of $\mathbf{U}_T$ and its n-th tensor product representation V_n. In Section 5 we construct the quiver Γ_n and the t-graded DG algebras R_n, $A \boxtimes R_n$ and show that they are formal as DG algebras. In Section 6 we define the t-graded DG $(H(R_n), A \boxtimes R_n)$-bimodule C_n. In Section 7 we finally give a categorification of the $\mathbf{U}_T$-action, $\mathcal{M}_n : H^0(DGP(A)) \times H^0(DGP(R_n)) \to H^0(DGP(R_n))$.

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2. $\mathbf{U}_T(\mathfrak{sl}(1|1))$ AND THE CATEGORIFICATION OF ITS MULTIPLICATION

In Section 2.1 we define the super Hopf algebra $\mathbf{U}_T$. In Section 2.2 we define the t-graded DG algebra A and the triangulated category $H^0(DGP(A))$ whose Grothendieck group is isomorphic to $\mathbf{U}_T$. In Section 2.3 we define the triangulated category $H^0(DGP(A \otimes A))$ whose Grothendieck group is isomorphic to $\mathbf{U}_T \otimes_{\mathbb{Z}[T^{\pm 1}]} \mathbf{U}_T$. In Section 2.4 we construct the t-graded DG $(A, A \otimes A)$-bimodule T. In Section 2.5 we categorify the multiplication $\mathbf{U}_T \otimes_{\mathbb{Z}[T^{\pm 1}]} \mathbf{U}_T \to \mathbf{U}_T$ to the exact functor $\eta : H^0(DGP(A \otimes A)) \xrightarrow{T\otimes_{A\otimes A} -} H^0(DGP(A))$.

2.1. **The super Hopf algebra $\mathbf{U}_T$.**

Definition 2.1. Define the super Hopf algebra $\{\mathbf{U}_T, m, \mathrm{p}, \Delta, \epsilon, S\}$ over $\mathbb{Z}$ as follows:

(1) The multiplication m makes $\mathbf{U}_T$ into an associative $\mathbb{Z}$-algebra with unit I, generators E, F, T, T^{-1} and relations:

$$E^2 = F^2 = 0,$$
$$EF + FE = I - T,$$
$$ET = TE,\ FT = TF,$$
$$TT^{-1} = T^{-1}T = I.$$

(2) The parity p is a $\mathbb{Z}$-grading[3] defined by:

$$\mathrm{p}(E) = -1, \quad \mathrm{p}(F) = 1, \quad \mathrm{p}(I) = \mathrm{p}(T) = 0.$$

(3) The comultiplication $\Delta : \mathbf{U}_T \to \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$ is an algebra map defined on the generators by:

$$\Delta(E) = E \otimes I + I \otimes E,$$
$$\Delta(F) = F \otimes T + I \otimes F,$$
$$\Delta(T) = T \otimes T.$$

(4) The counit $\epsilon : \mathbf{U}_T \to \mathbb{Z}$ is an algebra map defined on the generators by:

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(I) = \epsilon(T) = 1.$$

(5) The antipode $S : \mathbf{U}_T \to \mathbf{U}_T$ is an anti-homomorphism of superalgebras, i.e.,

$$S(ab) = (-1)^{\mathrm{p}(a)\,\mathrm{p}(b)} S(b)S(a),$$

defined on the generators by:

$$S(T) = T^{-1}, \quad S(E) = -E, \quad S(F) = -FT^{-1}.$$

Remark 2.2. (1) Since T is a central element, $\mathbf{U}_T$ can be viewed as a free $\mathbb{Z}[T^{\pm}]$-module over the basis $\mathcal{B} = \{I, E, F, EF\}$.
(2) The parity p actually comes from the Euler number of a dividing set. Recall a dividing set divides the surface into positive and negative regions. Then the *Euler number* is the Euler characteristic of the positive region minus the Euler characteristic of the negative region.
(3) The multiplication on $\mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$ is graded:

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\mathrm{p}(b)\,\mathrm{p}(c)} ac \otimes bd.$$

[3] In fact, this is a "categorical parity" rather than the usual parity taking values in $\mathbb{F}_2$.

(4) The counit corresponds to a functor $\tilde{\mathcal{C}}_o \to \tilde{\mathcal{C}}(S^2)$ between the contact categories of an annulus S_o and a sphere S^2 which is given by capping each component of ∂S_o off with a disk.
(5) The antipode corresponds to a functor $\tilde{\mathcal{C}}_o \to \tilde{\mathcal{C}}_o$ given by an inversion about the core of the annulus.

Lemma 2.3. *The definition above gives a super Hopf algebra* $\{\mathbf{U}_T, m, \mathrm{p}, \Delta, \epsilon, S\}$*:*

(1) Δ *is an algebra map.*
(2) Δ *is coassociative:* $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$.
(3) S *is an antipode:* $m \circ (S \otimes id) \circ \Delta(a) = m \circ (id \otimes S) \circ \Delta(a) = \epsilon(a) I$ *for all* $a \in \mathbf{U}_T$.

Proof. We verify (1) and leave (2) and (3) to the reader.

$$\begin{aligned}\Delta(E)\Delta(E) &= (E \otimes I + I \otimes E)(E \otimes I + I \otimes E) \\ &= E^2 \otimes I + I \otimes E^2 + (E \otimes I)(I \otimes E) + (I \otimes E)(E \otimes I) \\ &= E \otimes E - E \otimes E = 0.\end{aligned}$$

Similarly, $\Delta(F)\Delta(F) = 0$.

$$\begin{aligned}&\Delta(E)\Delta(F) + \Delta(F)\Delta(E) \\ =&(E \otimes I + I \otimes E)(F \otimes T + I \otimes F) + (F \otimes T + I \otimes F)(E \otimes I + I \otimes E) \\ =&(EF \otimes T - F \otimes ET + E \otimes F + I \otimes EF) + (FE \otimes T + F \otimes TE - E \otimes F + I \otimes FE) \\ =&(EF + FE) \otimes T + I \otimes (EF + FE) \\ =&I \otimes I - T \otimes T = \Delta(I - T) = \Delta(EF + FE). \quad \square\end{aligned}$$

2.2. **The t-graded DG algebra A.** We refer to [2, Section 10] for an introduction to DG algebras, DG modules and *projective* DG modules, and to [16] for an introduction to DG categories and their homology categories. A t-graded DG algebra R is a DG algebra with an additional t-grading. Let $DG(R)$ denote the DG category of t-graded DG left R-modules. We refer to [36] for more detail.

Definition 2.4. Let A be a t-graded DG $\mathbb{F}_2$-algebra with idempotents $e(\Gamma)$ for $\Gamma \in \mathcal{B} = \{F, I, EF, E\}$, generators $\rho(I, EF), \rho(EF, I)$ and relations:

$$e(\Gamma) \cdot e(\Gamma') = \delta_{\Gamma,\Gamma'} e(\Gamma) \text{ for } \Gamma, \Gamma' \in \mathcal{B};$$
$$e(I) \cdot \rho(I, EF) = \rho(I, EF) \cdot e(EF) = \rho(I, EF);$$
$$e(EF) \cdot \rho(EF, I) = \rho(EF, I) \cdot e(I) = \rho(EF, I);$$
$$\rho(I, EF) \cdot \rho(EF, I) = 0.$$

The differential on A is trivial. The grading $\deg = (\deg_h, \deg_t)$ is defined by:

$$\deg(a) = \begin{cases} (1,1) & \text{if } a = \rho(EF, I), \\ (0,0) & \text{otherwise}, \end{cases}$$

where the first component $\deg_h$ is the cohomological grading and the second component $\deg_t$ is the t-grading.

There is a decomposition of the algebra: $A = A_1 \oplus A_0 \oplus A_{-1}$, where A_1 is generated by $e(F)$; A_0 is generated by $e(I), e(EF), \rho(I, EF)$ and $\rho(EF, I)$; and A_{-1} is generated by $e(E)$.

Definition 2.5. The *parity* p is a $\mathbb{Z}$-grading on A defined by $\mathrm{p}(a) = i$ for $a \in A_i$.

Note that the parity p is not a grading with respect to the multiplication on A.

Remark 2.6. The algebra A can be viewed as a quotient of the path algebra $\mathbb{F}_2 Q_A$ of a quiver $Q_A = (V(Q_A), A(Q_A))$, where the vertex set $V(Q_A) = \mathcal{B} = \{F, I, EF, E\}$, and the arrow set $A(Q_A) = \{\rho(I, EF), \rho(EF, I)\}$. The quiver Q_A has three components according to the parity p on A.

Consider a collection of projective DG A-modules $\{P(\Gamma) = A \cdot e(\Gamma) \mid \Gamma \in \mathcal{B}\}$. As a left A-module, $P(\Gamma)$ is generated by the idempotent $e(\Gamma)$. In order to distinguish this generator from the idempotent in A, let $m(\Gamma)$ denote the generator of $P(\Gamma)$.

Definition 2.7. The *parity* p is a $\mathbb{Z}$-grading on the module $P(\Gamma)$ defined by $\mathrm{p}(m) = \mathrm{p}(\Gamma)$ for $m \in P(\Gamma)$ and $\Gamma \in \mathcal{B} \subset \mathbf{U}_T$, where $\mathrm{p}(\Gamma)$ is the parity of Γ in $\mathbf{U}_T$ from Definition 2.1.

Definition 2.8. Let $DGP(A)$ be the smallest full subcategory of $DG(A)$ which contains the projective DG A-modules $\{P(\Gamma) = A \cdot e(\Gamma) \mid \Gamma \in \mathcal{B}\}$ and is closed under the cohomological grading shift functor $[1]$, the t-grading shift functor $\{1\}$ and taking mapping cones.

The 0-th homology category $H^0(DGP(A))$ of $DGP(A)$ is the homotopy category of t-graded DG projective A-modules generated by $\{P(\Gamma) \mid \Gamma \in \mathcal{B}\}$. It is a triangulated category and the Grothendieck group $K_0(H^0(DGP(A)))$ has a $\mathbb{Z}[T^{\pm 1}]$-basis $\{P(\Gamma) \mid \Gamma \in \mathcal{B}\}$, where the multiplication by T is induced by the t-grading shift: $[M\{1\}] = T[M] \in K_0(H^0(DGP(A)))$ for $M \in H^0(DGP(A))$.

Lemma 2.9. *There is an isomorphism* $K_0(H^0(DGP(A))) \cong \mathbf{U}_T$ *of free* $\mathbb{Z}[T^{\pm 1}]$*-modules.*

2.3. The t-graded DG algebra $A \otimes A$.

Definition 2.10. Let $A \otimes A$ be the tensor product of two A's over $\mathbb{F}_2$ as an algebra. The differential is trivial. The grading $\deg = (\deg_h, \deg_t)$ is defined by:

$$\deg_t(a \otimes b) = \deg_t(a) + \deg_t(b),$$
$$\deg_h(a \otimes b) = \deg_h(a) + \deg_h(b) + 2\deg_t(a)\,\mathrm{p}(b),$$

for $a, b \in A$.

Remark 2.11. The cohomological grading $\deg_h$ of $A \otimes A$ is the sum of two $\deg_h$'s twisted by the t-grading $\deg_t$ and the parity p.

Consider a collection of projective DG $A \otimes A$-modules:

$$\{P(\Gamma, \Gamma') = (A \otimes A) \cdot (e(\Gamma) \otimes e(\Gamma')) \mid \Gamma, \Gamma' \in \mathcal{B}\}.$$

Definition 2.12. Let $DGP(A \otimes A)$ be the smallest full subcategory of $DG(A \otimes A)$ which contains the projective DG $A \otimes A$-modules $\{P(\Gamma, \Gamma') \mid \Gamma, \Gamma' \in \mathcal{B}\}$ and is closed under the cohomological grading shift functor $[1]$, the t-grading shift functor $\{1\}$ and taking mapping cones.

The 0-th homology category $H^0(DGP(A \otimes A))$ of $DGP(A \otimes A)$ is the homotopy category of t-graded DG projective $A \otimes A$-modules generated by $\{P(\Gamma, \Gamma') \mid \Gamma, \Gamma' \in \mathcal{B}\}$.

Definition 2.13. Define a tensor product functor

$$\begin{array}{rcccccc} \chi: & H^0(DGP(A)) & \times & H^0(DGP(A)) & \to & H^0(DGP(A \otimes A)) \\ & M_1 & , & M_2 & \mapsto & M_1 \otimes_{\mathbb{F}_2} M_2, \end{array}$$

where the grading on $M_1 \otimes M_2$ is given by:

$$\deg_t(m_1 \otimes m_2) = \deg_t(m_1) + \deg_t(m_2),$$
$$\deg_h(m_1 \otimes m_2) = \deg_h(m_1) + \deg_h(m_2) + 2\deg_t(m_1)\,\mathrm{p}(m_2),$$

for $m_1 \in M_1$ and $m_2 \in M_2$.

Remark 2.14. The grading in the tensor product functor χ is given by:

$$\chi(P(\Gamma)\{n\}, P(\Gamma')\{n'\}) = P(\Gamma, \Gamma')\{n + n'\}[2n\,\mathrm{p}(\Gamma')].$$

It is easy to see that the grading on $M_1 \otimes M_2$ makes it into a t-graded DG $A \otimes A$-module. Since $K_0(H^0(DGP(A \otimes A)))$ has a $\mathbb{Z}[T^{\pm 1}]$-basis $\mathcal{B} \times \mathcal{B}$, we have the following:

Lemma 2.15. *There is an isomorphism $K_0(H^0(DGP(A\otimes A))) \cong \mathbf{U}_T \otimes_{\mathbb{Z}[T^{\pm}]} \mathbf{U}_T$ of free $\mathbb{Z}[T^{\pm 1}]$-modules. Moreover, the functor χ induces a tensor product on their Grothendieck groups:*

$$K_0(\chi) : \mathbf{U}_T \times \mathbf{U}_T \xrightarrow{\otimes_{\mathbb{Z}[T^{\pm}]}} \mathbf{U}_T \otimes_{\mathbb{Z}[T^{\pm}]} \mathbf{U}_T.$$

2.4. **The t-graded DG $(A, A\otimes A)$-bimodule T.** To define a functor $\eta : DGP(A\otimes A) \to DGP(A)$ lifting the multiplication on $\mathbf{U}_T$, we define a DG $(A, A\otimes A)$-bimodule T in two steps: we define a left DG A-module T in Section 2.4.1 and a right DG $A\otimes A$-module structure on T in Section 2.4.2.

In practice, the functor η is obtained by "reverse-engineering": we have all the essential information about η from the contact topology and we construct the bimodule T to realize the functor algebraically. More precisely, we first figure out $\eta(P(\Gamma_1, \Gamma_2))$ and set

$$T = \bigoplus_{\Gamma_1,\Gamma_2 \in \mathcal{B}} \eta(P(\Gamma_1, \Gamma_2)) \in DGP(A).$$

as left DG A-modules. Then consider morphism sets $\mathrm{Hom}(P(\Gamma_1, \Gamma_2), P(\Gamma_1', \Gamma_2'))$ in $DGP(A\otimes A)$. For instance, the right multiplication by $e(F) \otimes \rho(I, EF)$ in $A \otimes A$ gives a specific morphism:

$$\begin{array}{rccc} \times(e(F) \otimes \rho(I, EF)) : & P(F, I) & \to & P(F, EF) \\ & m & \mapsto & m \cdot (e(F) \otimes \rho(I, EF)). \end{array}$$

The right multiplication on T by $e(F) \otimes \rho(I, EF)$ is given by the morphism

$$\eta(\times(e(F) \otimes \rho(I, EF))) : \eta(P(F, I)) \to \eta(P(F, EF))$$

in $DGP(A)$, where $\eta(P(F, I))$ and $\eta(P(F, EF))$ are viewed as left A-submodules of T. Note that this technique will be used to construct various bimodules in the paper.

2.4.1. *The left DG A-module T.*

Definition 2.16. Define a left DG A-module

$$T = \bigoplus_{\Gamma_1,\Gamma_2 \in \mathcal{B}} T(\Gamma_1, \Gamma_2),$$

where $(T(\Gamma_1, \Gamma_2), d(\Gamma_1, \Gamma_2)) \in DGP(A)$ is defined on a case-by-case basis as follows:

$$T(E, E) = T(E, EF) = T(F, F) = T(EF, F) = 0,$$

$$T(I, \Gamma) = T(\Gamma, I) = P(\Gamma) \text{ for all } \Gamma \in \mathcal{B},$$

$$T(E, F) = P(EF),$$

$$T(F, EF) = P(F) \oplus P(F)\{1\}[1],$$

$$T(EF,E) = P(E) \oplus P(E)\{1\}[-1],$$
$$T(EF,EF) = P(EF) \oplus P(EF)\{1\}[1],$$
$$T(F,E) = P(I) \oplus P(EF)[-1] \oplus P(I)\{1\}[-1].$$

$d(\Gamma_1,\Gamma_2) = 0$ for all $(\Gamma_1,\Gamma_2) \neq (F,E)$ and $d(F,E)$ is a map of left A-modules defined on generators of $T(F,E)$ by

$$\begin{array}{rccc} d(F,E): & T(F,E) & \to & T(F,E) \\ & m_{F,E}(I) & \mapsto & \rho(I,EF)\cdot m_{F,E}(EF), \\ & m_{F,E}(EF) & \mapsto & \rho(EF,I)\cdot m'_{F,E}(I), \\ & m'_{F,E}(I) & \mapsto & 0, \end{array}$$

where $m_{F,E}(I) \in P(I), m_{F,E}(EF) \in P(EF)[-1]$ and $m'_{F,E}(I) \in P(I)\{1\}[-1]$.

Remark 2.17. (1) The left DG A-module $T(\Gamma_1,\Gamma_2)$ is supposed to be the categorical multiplication of two DG A-modules $P(\Gamma_1)$ and $P(\Gamma_2)$. In particular, the class $[T(\Gamma_1,\Gamma_2)] \in \mathbf{U}_T$ is the multiplication $\Gamma_1 \cdot \Gamma_2 \in \mathbf{U}_T$ under the isomorphism in Lemma 2.9.
(2) In the contact category $\tilde{\mathcal{C}}_0$, the stacking $EF \cdot E$ of dividing sets is the union of E and a pair of loops. The pair of loops corresponds to tensoring with $\mathbb{Z}^2$ up to grading. Hence, the left A-module $T(EF,E)$ is given by a direct sum of two $P(E)$'s.
(3) The definition of $T(F,E)$ comes from a projective resolution of the left A-module which corresponds to the dividing set FE in the contact category $\tilde{\mathcal{C}}_0$.

Lemma 2.18. *$(T(F,E), d(F,E))$ is a t-graded DG A-module.*

Proof. It suffices to prove that $d = d(F,E)$ is of degree $(1,0)$ such that $d^2 = 0$. We verify that

$$d^2(m_{F,E}(I)) = d(\rho(I,EF)\cdot m_{F,E}(EF)) = \rho(I,EF)\cdot\rho(EF,I)\cdot m'_{F,E}(I) = 0.$$

The degrees of the generators of $T(F,E)$ are as follows:

$$\deg(m_{F,E}(I)) = (0,0), \quad \deg(m_{F,E}(EF)) = (1,0), \quad \deg(m'_{F,E}(I)) = (1,-1).$$

Hence,

$$\deg(d(m_{F,E}(I))) = \deg(\rho(I,EF)) + \deg(m_{F,E}(EF)) = (1,0) = \deg(m_{F,E}(I)) + (1,0);$$
$$\deg(d(m_{F,E}(EF))) = \deg(\rho(EF,I)) + \deg(m'_{F,E}(I)) = (2,0) = \deg(m_{F,E}(EF)) + (1,0);$$

which implies that the differential d is of degree $(1,0)$. □

2.4.2. *The right $A \otimes A$-module structure on T.* In this subsection we describe the right $A \otimes A$-module structure on T. Let $m \times (a \otimes b)$ denote the right multiplication for $m \in T, a \otimes b \in A \otimes A$ and let $m \cdot a$ denote the multiplication in A for $m \in P(\Gamma) \subset A, a \in A$.

We fix the notation for generators of $T(\Gamma_1, \Gamma_2)$:

$$\begin{gathered}
m_{\Gamma,I}(\Gamma) \in P(\Gamma) = T(\Gamma, I) \text{ for all } \Gamma \in \mathcal{B}; \\
m_{I,\Gamma}(\Gamma) \in P(\Gamma) = T(I, \Gamma) \text{ for all } \Gamma \in \mathcal{B}; \\
m_{E,F}(EF) \in P(EF) = T(E,F); \\
m_{EF,E}(E) \in P(E) \subset T(EF,E), \quad m'_{EF,E}(E) \in P(E)\{1\}[-1] \subset T(EF,E); \\
m_{F,EF}(F) \in P(F) \subset T(F,EF), \quad m'_{F,EF}(F) \in P(F)\{1\}[1] \subset T(F,EF); \\
m_{EF,EF}(EF) \in P(EF) \subset T(EF,EF), \quad m'_{EF,EF}(EF) \in P(EF)\{1\}[1] \subset T(EF,EF).
\end{gathered}$$

We define the right multiplication on a case-by-case basis as follows:

(1) For an idempotent $e(\Gamma_1) \otimes e(\Gamma_2)$, define

$$\begin{array}{rccc}
\times(e(\Gamma_1) \otimes e(\Gamma_2)): & T(\Gamma'_1, \Gamma'_2) & \to & T(\Gamma'_1, \Gamma'_2) \\
& m & \mapsto & \delta_{\Gamma_1,\Gamma'_1}\delta_{\Gamma_2,\Gamma'_2}\, m
\end{array}$$

(2) For generators $\rho(EF,I) \otimes e(E)$ and $\rho(I,EF) \otimes e(E)$, define

$$\begin{array}{rccc}
\times(\rho(EF,I) \otimes e(E)): & T(EF,E) & \to & T(I,E) \\
& m_{EF,E}(E) & \mapsto & 0 \\
& m'_{EF,E}(E) & \mapsto & m_{I,E}(E),
\end{array}$$

$$\begin{array}{rccc}
\times(\rho(I,EF) \otimes e(E)): & T(I,E) & \to & T(EF,E) \\
& m_{I,E}(E) & \mapsto & m_{EF,E}(E).
\end{array}$$

(3) For generators $e(F) \otimes \rho(EF,I)$ and $e(F) \otimes \rho(I,EF)$, define

$$\begin{array}{rccc}
\times(e(F) \otimes \rho(EF,I)): & T(F,EF) & \to & T(F,I) \\
& m_{F,EF}(F) & \mapsto & 0 \\
& m'_{F,EF}(F) & \mapsto & m_{F,I}(F),
\end{array}$$

$$\begin{array}{rccc}
\times(e(F) \otimes \rho(I,EF)): & T(F,I) & \to & T(F,EF) \\
& m_{F,I}(F) & \mapsto & m_{F,EF}(F).
\end{array}$$

(4) For generators $e(I) \otimes a$ and $a \otimes e(I)$, where $a \in \{\rho(I,EF), \rho(EF,I)\}$, define

$$\begin{array}{rccc}
\times(\rho(EF,I) \otimes e(I)): & T(EF,I) & \to & T(I,I) \\
& m_{EF,I}(EF) & \mapsto & \rho(EF,I) \cdot m_{I,I}(I),
\end{array}$$

$$\begin{array}{rccc}
\times(\rho(I,EF)\otimes e(I)): & T(I,I) & \to & T(EF,I) \\
 & m_{I,I}(I) & \mapsto & \rho(I,EF)\cdot m_{EF,I}(EF), \\
\times(e(I)\otimes \rho(EF,I)): & T(I,EF) & \to & T(I,I) \\
 & m_{I,EF}(EF) & \mapsto & \rho(EF,I)\cdot m_{I,I}(I), \\
\times(e(I)\otimes \rho(I,EF)): & T(I,I) & \to & T(I,EF) \\
 & m_{I,I}(I) & \mapsto & \rho(I,EF)\cdot m_{I,EF}(EF).
\end{array}$$

(5) For generators $e(EF)\otimes a$ and $a\otimes e(EF)$, where $a\in\{\rho(I,EF),\rho(EF,I)\}$, define

$$\begin{array}{rccc}
\times(\rho(EF,I)\otimes e(EF)): & T(EF,EF) & \to & T(I,EF) \\
 & m_{EF,EF}(EF) & \mapsto & 0 \\
 & m'_{EF,EF}(EF) & \mapsto & m_{I,EF}(EF), \\
\times(e(EF)\otimes \rho(EF,I)): & T(EF,EF) & \to & T(EF,I) \\
 & m_{EF,EF}(EF) & \mapsto & 0 \\
 & m'_{EF,EF}(EF) & \mapsto & m_{EF,I}(EF), \\
\times(\rho(I,EF)\otimes e(EF)): & T(I,EF) & \to & T(EF,EF) \\
 & m_{I,EF}(EF) & \mapsto & m_{EF,EF}(EF), \\
\times(e(EF)\otimes \rho(I,EF)): & T(EF,I) & \to & T(EF,EF) \\
 & m_{EF,I}(EF) & \mapsto & m_{EF,EF}(EF).
\end{array}$$

(6) For generators $e(E)\otimes\rho(EF,I), e(E)\otimes\rho(I,EF), \rho(EF,I)\otimes e(F)$ and $\rho(I,EF)\otimes e(F)$, define the right multiplication to be the zero map since the corresponding domains or ranges are trivial from Definition 2.16.

Remark 2.19. The definition above is compatible with the grading on T:

$$\deg(m\times r)=\deg(m)+\deg(r).$$

This concludes the definition of the right $A\otimes A$-module structure on T. We need to show that the definition above is compatible with the relations of $A\otimes A$:

$$(m\times r_1)\times r_2=(m\times r_1')\times r_2'$$

for $r_1\cdot r_2=r_1'\cdot r_2'\in A\otimes A$. This can be verified case by case and we leave it to the reader.

It is easy to see that the left A-module structure and the right $A\otimes A$-module structure on T are compatible: $a\cdot(m\times r)=(a\cdot m)\times r$, for $a\in A, r\in A\otimes A$ and $m\in T$. Hence T is a t-graded DG $(A,A\otimes A)$-bimodule.

2.5. **The categorification of the multiplication on $\mathbf{U}_T$.** In this section, we use the bimodule T to categorify the multiplication on $\mathbf{U}_T$, i.e., prove Theorem 1.1.

Definition 2.20. Let $\eta : DGP(A \otimes A) \xrightarrow{T \otimes_{A\otimes A} -} DGP(A)$ be a functor of tensoring with the DG $(A, A \otimes A)$-bimodule T over $A \otimes A$.

Lemma 2.21. *The functor η maps $P(\Gamma_1, \Gamma_2)$ to $T(\Gamma_1, \Gamma_2) \in DGP(A)$ for all $\Gamma_1, \Gamma_2 \in \mathcal{B}$.*

Proof. Since $T = \bigoplus\limits_{\Gamma'_1,\Gamma'_2\in\mathcal{B}} T(\Gamma'_1, \Gamma'_2)$ as left DG A-modules, it follows that $T \otimes P(\Gamma_1, \Gamma_2)$ is the quotient of

$$\bigoplus_{\Gamma'_1,\Gamma'_2\in\mathcal{B}} (T(\Gamma'_1, \Gamma'_2) \times P(\Gamma_1, \Gamma_2))$$

by the relations

$$\{(m \times r, e(\Gamma_1, \Gamma_2)) = (m, r \cdot e(\Gamma_1, \Gamma_2)) \mid m \in T(\Gamma'_1, \Gamma'_2), r \in A \otimes A\}.$$

Since $T(\Gamma'_1, \Gamma'_2) \times P(\Gamma_1, \Gamma_2)$ is spanned by $\{(m, r \cdot e(\Gamma_1, \Gamma_2)) \mid m \in T(\Gamma'_1, \Gamma'_2), r \cdot e(\Gamma_1, \Gamma_2) \neq 0\}$, $T \otimes P(\Gamma_1, \Gamma_2)$ is spanned by

$$\{(m \times r, e(\Gamma_1, \Gamma_2)) \mid m \in T(\Gamma'_1, \Gamma'_2), r \cdot e(\Gamma_1, \Gamma_2) \neq 0\} \cong T(\Gamma_1, \Gamma_2) \in DGP(A). \quad \square$$

There is an induced exact functor η between the 0-th homology categories:

$$\eta : H^0(DGP(A \otimes A)) \xrightarrow{T \otimes_{A\otimes A} -} H^0(DGP(A)).$$

Let $\mathcal{M} = \eta \circ \chi$ be the composition:

$$\mathcal{M} : H^0(DGP(A)) \times H^0(DGP(A)) \xrightarrow{\chi} H^0(DGP(A \otimes A)) \xrightarrow{\eta} H^0(DGP(A)).$$

Proof of Theorem 1.1. We compute the multiplication

$$K_0(\mathcal{M}) : K_0(H^0(DGP(A))) \times K_0(H^0(DGP(A))) \to K_0(H^0(DGP(A))).$$

(1) By Lemma 2.21, $\mathcal{M}(P(\Gamma), P(\Gamma')) = \eta(P(\Gamma, \Gamma')) = T(\Gamma, \Gamma')$, for $\Gamma, \Gamma' \in \mathcal{B}$. Its class $[T(\Gamma, \Gamma')]$ agrees with $\Gamma \cdot \Gamma' \in \mathbf{U}_T$ by Remark 2.17.

(2) The class $[P(I)]$ is a unit of $K_0(H^0(DGP(A)))$, since $P(I)$ is a unit under $\mathcal{M}$:

$$\mathcal{M}(P(\Gamma), P(I)) = \eta(P(\Gamma, I)) = T(\Gamma, I) = P(\Gamma),$$
$$\mathcal{M}(P(I), P(\Gamma)) = \eta(P(I, \Gamma)) = T(I, \Gamma) = P(\Gamma).$$

(3) By Remark 2.14,

$$\mathcal{M}(P(\Gamma), P(I)\{1\}) = \eta(P(\Gamma, I)\{1\}) = P(\Gamma)\{1\};$$
$$\mathcal{M}(P(I)\{1\}, P(\Gamma)) = \eta(P(I, \Gamma)\{1\}[2\,\mathrm{p}(\Gamma)]) = P(\Gamma)\{1\}[2\,\mathrm{p}(\Gamma)].$$

Although $\mathcal{M}(P(\Gamma), P(I)\{1\})$ and $\mathcal{M}(P(I)\{1\}, P(\Gamma))$ differ by $2\,\mathrm{p}(\Gamma)$ in their cohomological gradings, their classes agree in $K_0(H^0(DGP(A)))$. Hence $[P(I)\{1\}]$ corresponds to the variable T in the $\mathbb{Z}[T^{\pm 1}]$-algebra $K_0(H^0(DGP(A)))$.

(1), (2) and (3) together imply that the following map is an isomorphism of $\mathbb{Z}[T^{\pm 1}]$-algebras:

$$\begin{array}{ccc} \mathbf{U}_T & \to & K_0(H^0(DGP(A))) \\ \Gamma & \mapsto & [P(\Gamma)], \\ T & \mapsto & [P(I)\{1\}]. \quad \square \end{array}$$

Remark 2.22. It is natural to ask whether $\mathcal{M}$ is a monoidal functor, i.e., the following diagram commutes up to equivalence:

$$\begin{tikzcd} H^0(DGP(A)) \times H^0(DGP(A)) \times H^0(DGP(A)) \arrow[r, "id\times\mathcal{M}"] \arrow[d, "\mathcal{M}\times id"] & H^0(DGP(A)) \times H^0(DGP(A)) \arrow[d, "\mathcal{M}"] \\ H^0(DGP(A)) \times H^0(DGP(A)) \arrow[r, "\mathcal{M}"] & H^0(DGP(A)). \end{tikzcd}$$

We believe that the answer is positive and it could be done by verifying some associativity relation on various DG bimodules.

3. THE CATEGORIFICATION OF THE COMULTIPLICATION ON $\mathbf{U}_T(\mathfrak{sl}(1|1))$

To categorify the comultiplication $\Delta : \mathbf{U}_T \to \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$, we define the (t_1, t_2)-graded DG algebra B and the triangulated category $H^0(DGP(B))$ whose Grothendieck group is isomorphic to $\mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$. Then we construct the (t_1, t_2)-graded DG (B, A)-bimodule S to give an exact functor $\delta : H^0(DGP(A)) \xrightarrow{S\otimes_A -} H^0(DGP(B))$. The decategorification $K_0(\delta)$ on the Grothendieck groups agrees with the comultiplication on $\mathbf{U}_T$.

3.1. **The (t_1, t_2)-graded DG algebra B.** We define the algebra B via a quiver Q_B.

3.1.1. *The quiver Q_B.*

Definition 3.1 (Quiver $Q_B = (V(Q_B), A(Q_B))$)**.** Let $V(Q_B) = \mathcal{B} \times \mathcal{B}$ be the set of vertices. Let $A(Q_B)$ be the set of arrows consisting of 3 groups:

$$\begin{aligned}
A(Q_{B,-1}) = \{&E \otimes I \to E \otimes EF,\ E \otimes EF \to E \otimes I,\ E \otimes I \to I \otimes E,\ I \otimes E \to EF \otimes E,\\
&EF \otimes E \to I \otimes E\};\\
A(Q_{B,0}) = \{&E \otimes F \to I \otimes EF,\ I \otimes EF \to I \otimes I,\ I \otimes I \to I \otimes EF,\ I \otimes EF \to EF \otimes EF,\\
&EF \otimes EF \to EF \otimes I,\ EF \otimes I \to I \otimes I,\ I \otimes I \to EF \otimes I,\ EF \otimes I \to F \otimes E,\\
&EF \otimes I \to EF \otimes EF,\ EF \otimes EF \to I \otimes EF\};\\
A(Q_{B,1}) = \{&I \otimes F \to EF \otimes F,\ EF \otimes F \to I \otimes F,\ I \otimes F \to F \otimes I,\ F \otimes I \to F \otimes EF,\\
&F \otimes EF \to F \otimes I\}.
\end{aligned}$$

$$\begin{tikzcd}
E \otimes EF \arrow[d, shift left] & EF \otimes E \arrow[d, shift left] \\
E \otimes I \arrow[u, shift left] \arrow[r] & I \otimes E \arrow[u, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}
E \otimes F \arrow[r] & I \otimes EF \arrow[r, shift left] \arrow[d, shift left] & I \otimes I \arrow[l, shift left] \arrow[d, shift left] \\
& EF \otimes EF \arrow[u, shift left] \arrow[r, shift left] & EF \otimes I \arrow[l, shift left] \arrow[u, shift left] \arrow[d] \\
& & F \otimes E
\end{tikzcd}$$

$$\begin{tikzcd}
EF \otimes F \arrow[d, shift left] & F \otimes EF \arrow[d, shift left] \\
I \otimes F \arrow[u, shift left] \arrow[r] & F \otimes I \arrow[u, shift left]
\end{tikzcd}$$

Remark 3.2. (1) The quiver has 5 components $Q_B = \bigoplus_{i=-2}^{2} Q_{B,i}$ according to the parity p on $\mathcal{B}$, i.e., vertices $\{\Gamma_1 \otimes \Gamma_2 \in \mathcal{B} \times \mathcal{B}\}$ in $Q_{B,i}$ satisfy $\mathrm{p}(\Gamma_1) + \mathrm{p}(\Gamma_2) = i$. The diagrams on the left are the components $Q_{B,-1}$ and $Q_{B,1}$. The diagram on the right is the component $Q_{B,0}$. There is no arrow in $Q_{B,-2}$ and $Q_{B,2}$.
(2) Each arrow corresponds to a tight contact structure in $S_{oo} \times I$ with certain dividing sets determined by two vertices of the arrow.

3.1.2. *The (t_1, t_2)-graded DG algebra B.*

We define the (t_1, t_2)-graded algebra $B = \bigoplus_{i=-2}^{2} B_i$, where B_i is a quotient of the path algebra $\mathbb{F}_2 Q_{B,i}$ of the quiver $Q_{B,i}$.

Definition 3.3. B is an associative (t_1, t_2)-graded $\mathbb{F}_2$-algebra with a trivial differential and a grading $\deg = (\deg_h; \deg_{t_1}, \deg_{t_2}) \in \mathbb{Z}^3$.

(1) B has idempotents $e(\Gamma_1 \otimes \Gamma_2)$ for all vertices $\Gamma_1 \otimes \Gamma_2 \in \mathcal{B} \times \mathcal{B}$, generators $\rho(\Gamma_1 \otimes \Gamma_2, \Gamma_1' \otimes \Gamma_2')$ for all arrows $\Gamma_1 \otimes \Gamma_2 \to \Gamma_1' \otimes \Gamma_2'$ in Q_B. The relations consists of 4 groups:
(i) idempotents:

$$e(\Gamma_1 \otimes \Gamma_2) \cdot e(\Gamma_1' \otimes \Gamma_2') = \delta_{\Gamma_1,\Gamma_1'} \delta_{\Gamma_2,\Gamma_2'} \, e(\Gamma_1 \otimes \Gamma_2) \text{ for } \Gamma_1, \Gamma_2, \Gamma_1', \Gamma_2' \in \mathcal{B},$$
$$e(\Gamma_1 \otimes \Gamma_2) \cdot \rho(\Gamma_1 \otimes \Gamma_2, \Gamma_1' \otimes \Gamma_2') = \rho(\Gamma_1 \otimes \Gamma_2, \Gamma_1' \otimes \Gamma_2') \cdot e(\Gamma_1' \otimes \Gamma_2') = \rho(\Gamma_1 \otimes \Gamma_2, \Gamma_1' \otimes \Gamma_2');$$

(ii) relations in B_{-1} of 2 groups:
(A) relations from the algebra A:

$$\rho(E \otimes I, E \otimes EF) \cdot \rho(E \otimes EF, E \otimes I) = 0,$$
$$\rho(I \otimes E, EF \otimes E) \cdot \rho(EF \otimes E, I \otimes E) = 0;$$

(B) relations for $\rho(E \otimes I, I \otimes E)$:

$$\rho(E \otimes EF, E \otimes I) \cdot \rho(E \otimes I, I \otimes E) = 0,$$
$$\rho(E \otimes I, I \otimes E) \cdot \rho(I \otimes E, EF \otimes E) = 0;$$

(iii) relations in B_0 of 3 groups:
(A) relations from the algebra A:

$$\rho(I \otimes I, I \otimes EF) \cdot \rho(I \otimes EF, I \otimes I) = 0,$$
$$\rho(I \otimes I, EF \otimes I) \cdot \rho(EF \otimes I, I \otimes I) = 0,$$
$$\rho(I \otimes EF, EF \otimes EF) \cdot \rho(EF \otimes EF, I \otimes EF) = 0,$$
$$\rho(EF \otimes I, EF \otimes EF) \cdot \rho(EF \otimes EF, EF \otimes I) = 0;$$

(B) commutativity relations:

$$\rho(I \otimes I, I \otimes EF) \cdot \rho(I \otimes EF, EF \otimes EF) = \rho(I \otimes I, EF \otimes I) \cdot \rho(EF \otimes I, EF \otimes EF),$$
$$\rho(I \otimes EF, I \otimes I) \cdot \rho(I \otimes I, EF \otimes I) = \rho(I \otimes EF, EF \otimes EF) \cdot \rho(EF \otimes EF, EF \otimes I),$$
$$\rho(EF \otimes I, I \otimes I) \cdot \rho(I \otimes I, I \otimes EF) = \rho(EF \otimes I, EF \otimes EF) \cdot \rho(EF \otimes EF, I \otimes EF),$$
$$\rho(EF \otimes EF, I \otimes EF) \cdot \rho(I \otimes EF, I \otimes I) = \rho(EF \otimes EF, EF \otimes I) \cdot \rho(EF \otimes I, I \otimes I);$$

(C) relations for $E \otimes F$ and $F \otimes E$:

$$\rho(E \otimes F, I \otimes EF) \cdot \rho(I \otimes EF, EF \otimes EF) = 0,$$
$$\rho(EF \otimes EF, EF \otimes I) \cdot \rho(EF \otimes I, F \otimes E) = 0;$$

(iv) relations in B_1 of 2 groups:

(A) relations from the algebra A:

$$\rho(I \otimes F, EF \otimes F) \cdot \rho(EF \otimes F, I \otimes F) = 0,$$
$$\rho(F \otimes I, F \otimes EF) \cdot \rho(F \otimes EF, F \otimes I) = 0;$$

(B) relations for $\rho(I \otimes F, F \otimes I)$:

$$\rho(EF \otimes F, I \otimes F) \cdot \rho(I \otimes F, F \otimes I) = 0,$$
$$\rho(I \otimes F, F \otimes I) \cdot \rho(F \otimes I, F \otimes EF) = 0.$$

(2) The grading $\deg = (\deg_h; \deg_{t_1}, \deg_{t_2})$ is defined on the generators by:

$$\deg(a) = \begin{cases} (1;0,0) & \text{if } a = \rho(E \otimes I, I \otimes E),\ \rho(E \otimes F, I \otimes EF), \\ (1;1,0) & \text{if } a = \rho(EF \otimes \Gamma, I \otimes \Gamma) \text{ for all } \Gamma \in \mathcal{B}, \\ (1;0,1) & \text{if } a = \rho(I \otimes F, F \otimes I),\ \rho(\Gamma \otimes EF, \Gamma \otimes I) \text{ for all } \Gamma \in \mathcal{B}, \\ (0;0,0) & \text{otherwise}, \end{cases}$$

where $\deg_h$ is the cohomological grading and $(\deg_{t_1}, \deg_{t_2})$ is the (t_1, t_2)-grading.

Remark 3.4. (1) Each generator represents a tight contact structure given by one bypass attachment. Commutativity Relations (iii-B) come from certain isotopies of tight contact structures. Other relations come from the fact that gluing of the corresponding tight contact structures is not tight.

(2) Relations (ii-A),(iii-A), (iii-B) and (iv-A) come from the relation $\rho(I, EF) \cdot \rho(EF, I) = 0$ in A.

(3) The generators in $A \otimes A$ can be translated into some generators in B: $\rho(I, EF) \otimes e(I) \in A \otimes A$ corresponds to $\rho(I \otimes I, EF \otimes I) \in B$ for instance. But the gradings on $A \otimes A$ and B are quite different.

(4) The algebra B is actually the homology of the strands algebra for a specific handle decomposition of a twice punctured disk. We refer to Section 5.1 for more detail.

Definition 3.5. Let $DGP(B)$ be the smallest full subcategory of $DG(B)$ which contains the projective DG B-modules $\{P(\Gamma_1 \otimes \Gamma_2) = B \cdot e(\Gamma_1 \otimes \Gamma_2) \mid \Gamma_1, \Gamma_2 \in \mathcal{B}\}$ and is closed under the cohomological grading shift functor $[1]$, two (t_1, t_2)-grading shift functors $\{1, 0\}$ and $\{0, 1\}$, and taking mapping cones.

The 0-th homology category $H^0(DGP(B))$ of $DGP(B)$ is a triangulated category and the Grothendieck group $K_0(H^0(DGP(B)))$ has a $\mathbb{Z}[T^{\pm 1} \otimes T^{\pm 1}]$-basis

$$\{P(\Gamma_1 \otimes \Gamma_2) \mid \Gamma_1, \Gamma_2 \in \mathcal{B}\} \cong \mathcal{B} \times \mathcal{B},$$

where the multiplication by $T \otimes 1$ and $1 \otimes T$ are induced by the (t_1, t_2)-grading shifts:

$$[M\{1,0\}] = (T \otimes 1)[M], \quad [M\{0,1\}] = (1 \otimes T)[M] \in K_0(H^0(DGP(A))),$$

for $M \in H^0(DGP(B))$.

Lemma 3.6. *There is an isomorphism of free* $\mathbb{Z}[T^{\pm 1} \otimes T^{\pm 1}]$*-modules:*

$$K_0(H^0(DGP(B))) \cong \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T.$$

3.2. **The** (t_1, t_2)**-graded DG** (B, A)**-bimodule** S**.** To define a functor $\delta : DGP(A) \to DGP(B)$ lifting the comultiplication on $\mathbf{U}_T$, we construct a (t_1, t_2)-graded DG (B, A)-bimodule S in two steps: we first define a left DG B-module S in Section 3.2.1 and then define a right A-module structure on S in Section 3.2.2.

3.2.1. *The left* B*-module* S*.*

Definition 3.7. Define a (t_1, t_2)-graded left DG B-module

$$S = \bigoplus_{\Gamma \in \mathcal{B}} S(\Gamma),$$

where $(S(\Gamma), d(\Gamma)) \in DGP(B)$ is defined on a case-by-case basis as follows:
(1) $S(I) = P(I \otimes I); d(I) = 0.$
(2) $S(E) = P(E \otimes I) \oplus P(I \otimes E)$; $d(E)$ is a map of left B-modules defined on the generators by

$$\begin{array}{rccc} d(E): & S(E) & \to & S(E) \\ & m(E \otimes I) & \mapsto & \rho(E \otimes I, I \otimes E) \cdot m(I \otimes E), \\ & m(I \otimes E) & \mapsto & 0. \end{array}$$

(3) $S(F) = P(I \otimes F) \oplus P(F \otimes I)\{0,1\}$; $d(F)$ is a map of left B-modules defined on the generators by

$$\begin{array}{rccc} d(F): & S(F) & \to & S(F) \\ & m(I \otimes F) & \mapsto & \rho(I \otimes F, F \otimes I) \cdot m(F \otimes I), \\ & m(F \otimes I) & \mapsto & 0. \end{array}$$

(4) $S(EF) = P(E \otimes F) \oplus P(I \otimes EF) \oplus P(EF \otimes I)\{0,1\} \oplus P(F \otimes E)\{0,1\}[-1]$; $d(EF)$ is a map of left B-modules defined on the generators by

$$
\begin{array}{rrcc}
d(EF): & S(EF) & \to & S(EF) \\
& m(E\otimes F) & \mapsto & \rho(E\otimes F, I\otimes EF)\cdot m(I\otimes EF), \\
& m(I\otimes EF) & \mapsto & \rho(I\otimes EF, EF\otimes EF)\cdot \rho(EF\otimes EF, EF\otimes I)\cdot m(EF\otimes I), \\
& m(EF\otimes I) & \mapsto & \rho(EF\otimes I, F\otimes E)\cdot m(F\otimes E), \\
& m(F\otimes E) & \mapsto & 0.
\end{array}
$$

Remark 3.8. (1) The DG B-modules $S(\Gamma)$ are supposed to be the categorical comultiplication of the DG A-modules $P(\Gamma)$, for all $\Gamma\in\mathcal{B}$. In particular, the classes $[S(\Gamma)]\in K_0(H^0(DGP(B)))$ agree with the comultiplication $\Delta(\Gamma)\in \mathbf{U}_T\otimes_{\mathbb{Z}}\mathbf{U}_T$ under the isomorphism in Lemma 3.6.
(2) The definition of $S(\Gamma)$ comes from a projective resolution of the left B-module which corresponds to the stacking of Γ and $I\otimes I$ in the contact category $\tilde{\mathcal{C}}_{oo}$.

Lemma 3.9. *$(S(\Gamma), d(\Gamma))$ is a (t_1,t_2)-graded DG B-module.*

Proof. It suffices to prove that $d(\Gamma)$ is of degree $(1;0,0)$ such that $d(\Gamma)^2=0$. We verify it for $\Gamma=EF$ and leave other cases to the reader.

$$
\begin{aligned}
d^2(m(E\otimes F)) =& \rho(E\otimes F, I\otimes EF)\cdot\rho(I\otimes EF, EF\otimes EF)\cdot\rho(EF\otimes EF, EF\otimes I) \\
=& 0\cdot\rho(EF\otimes EF, EF\otimes I)=0, \\
d^2(m(I\otimes EF)) =& \rho(I\otimes EF, EF\otimes EF)\cdot\rho(EF\otimes EF, EF\otimes I)\cdot\rho(EF\otimes I, F\otimes E) \\
=& \rho(I\otimes EF, EF\otimes EF)\cdot 0=0,
\end{aligned}
$$

from Relation (iii-C) in Definition 3.7. $d^2=0$ is obvious for the other two generators of $S(EF)$.

The degrees of the generators of $S(EF)$ are as follows:

$$
\deg(m(E\otimes F))=\deg(m(I\otimes EF))=(0;0,0),
$$
$$
\deg(m(EF\otimes I))=(0;0,-1),\quad \deg(m(F\otimes E))=(1;0,-1).
$$

Hence,

$$
\begin{aligned}
\deg(d(m(E\otimes F))) =& \deg(\rho(E\otimes F, I\otimes EF))+\deg(m(I\otimes EF)) \\
=& (1;0,0)=\deg(m(E\otimes F))+(1;0,0), \\
\deg(d(m(EF\otimes I))) =& \deg(\rho(EF\otimes I, F\otimes E))+\deg(m(F\otimes E)) \\
=& (1;0,-1)=\deg(m(E\otimes F))+(1;0,0),
\end{aligned}
$$

$$\begin{aligned}\deg(d(m(I\otimes EF))) =& \deg(\rho(I\otimes EF, EF\otimes EF)) + \deg(\rho(EF\otimes EF, EF\otimes I)) \\ & + \deg(m(EF\otimes I)) = (1;0,0) = \deg(m(I\otimes EF)) + (1;0,0),\end{aligned}$$

which implies that the differential d is of degree $(1;0,0)$. □

3.2.2. *The right A-module structure on S.* In this subsection we describe the right A-module structure on S. Let $m\times a$ denote the right multiplication for $m\in S, a\in A$ and let $m\cdot b$ denote the multiplication in B for $m\in P(\Gamma_1\otimes\Gamma_2)\subset B, b\in B$.

Definition 3.10. For $m\in S$ and $a\in A$, define the grading of the right multiplication $m\times a$ by:

$$\deg(m\times a) = \deg(m) + (\deg_h(a); \deg_t(a), \deg_t(a)),$$

where $\deg$ is the grading in B, $\deg_h$ and $\deg_t$ are the gradings in A.

Remark 3.11. This definition is related to the categorification of $\Delta(T) = T\otimes T$ in the proof of Theorem 1.2 in Section 3.3.

The right multiplication is a map of left DG B-modules defined on generators as follows:
(1) For an idempotent $e(\Gamma)$, define

$$\begin{array}{cccc} \times e(\Gamma): & S(\Gamma') & \to & S(\Gamma') \\ & m & \mapsto & \delta_{\Gamma,\Gamma'}\, m. \end{array}$$

(2) For the generator $\rho(I, EF)$, define

$$\begin{array}{cccc} \times\rho(I,EF): & S(I) & \to & S(EF) \\ & m(I\otimes I) & \mapsto & \rho(I\otimes I, I\otimes EF)\cdot m(I\otimes EF). \end{array}$$

(3) For the generator $\rho(EF, I)$, define

$$\begin{array}{cccc} \times\rho(EF,I): & S(EF) & \to & S(I) \\ & m(EF\otimes I) & \mapsto & \rho(EF\otimes I, I\otimes I)\cdot m(I\otimes I), \\ & m(E\otimes F) & \mapsto & 0, \\ & m(I\otimes EF) & \mapsto & 0, \\ & m(F\otimes E) & \mapsto & 0. \end{array}$$

This concludes the definition of the right DG A-module structure on S.

Lemma 3.12. *The definition above gives a right DG A-module S:*

(1) $(m\times a_1)\times a_2 = (m\times a_1')\times a_2'$ *for* $a_1\cdot a_2 = a_1'\cdot a_2' \in A$ *and* $m\in S$.

(2) $d(m \times a) = d(m) \times a$ *for* $a \in A$ *and* $m \in M$.
(3) *The right multiplication is compatible with the grading in Definition 3.10.*

Proof. For (1), since the only non-trivial relation in A is $\rho(I, EF)) \cdot \rho(EF, I) = 0$, it suffices to prove that

$$(m \times \rho(I, EF)) \times \rho(EF, I) = 0,$$

which follows from the definition.

For (2), we verify that

$$\begin{aligned} d(m(I \otimes I) \times \rho(I, EF)) =& \rho(I \otimes I, I \otimes EF) \cdot d(m(I \otimes EF)) \\ =& \rho(I \otimes I, I \otimes EF) \cdot \rho(I \otimes EF, I \otimes I) \cdot \rho(I \otimes I, EF \otimes I) \\ =& 0 = d(m(I \otimes I)) \times \rho(I, EF), \end{aligned}$$

from Relations (iii-A) and (iii-B) in Definition 3.7. Similarly,

$$d(m(I \otimes EF)) \times \rho(EF, I) = 0 = d(m(I \otimes EF) \times \rho(EF, I)).$$

For (3), we verify that

$$\deg(m(EF \otimes I) \times \rho(EF, I)) = (0; 0, -1) + (1; 1, 1) = \deg(\rho(EF \otimes I, I \otimes I) \cdot m(I \otimes I)).$$

Similarly, $\deg(m(I \otimes I) \times \rho(I, EF)) = \deg(\rho(I \otimes I, I \otimes EF) \cdot m(I \otimes EF))$. □

It is easy to see that the left B-module structure and the right A-module structure on S are compatible: $b \cdot (m \times a) = (b \cdot m) \times a$, for $a \in A, b \in B$ and $m \in S$. Hence S is a (t_1, t_2)-graded DG (B, A)-bimodule.

3.3. **The categorification of the comultiplication on $\mathbf{U}_T$.** In this section, we use the bimodule S to categorify the comultiplication on $\mathbf{U}_T$, i.e., prove Theorem 1.2.

Definition 3.13. Let $\delta : DGP(A) \xrightarrow{S \otimes_A -} DGP(B)$ be a functor of tensoring with the DG (B, A)-bimodule S over A.

Lemma 3.14. *The functor* δ *maps* $P(\Gamma)$ *to* $S(\Gamma) \in DGP(B)$ *for all* $\Gamma \in \mathcal{B}$.

Proof. The proof is similar to that of Lemma 2.21. □

There is an induced exact functor δ between the 0-th homology categories:

$$\delta : H^0(DGP(A)) \xrightarrow{S \otimes_A -} H^0(DGP(B)).$$

Proof of Theorem 1.2. We compute the map on the Grothendieck groups:

$$K_0(\delta) : K_0(H^0(DGP(A))) \to K_0(H^0(DGP(B))).$$

(1) By Lemma 3.14, $\delta(P(\Gamma)) = S(\Gamma)$, for $\Gamma \in \mathcal{B} = \{I, E, F, EF\}$. Hence by Remark 3.8,

$$K_0(\delta)[P(\Gamma)] = [S(\Gamma)] = \Delta(\Gamma) \in \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T.$$

(2) By the grading of the right multiplication in Definition 3.10, $\delta(P(\Gamma)\{n\}) = S(\Gamma)\{n, n\}$ for $n \in \mathbb{Z}$. Hence,

$$K_0(\delta)(T^n[P(\Gamma)]) = K_0(\delta)([P(\Gamma)\{n\}]) = [S(\Gamma)\{n, n\}] = (T^n \otimes T^n)[S(\Gamma)]$$

.

(1) and (2) together imply that $K_0(\delta) = \Delta : \mathbf{U}_T \to \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$ since the $\mathbb{Z}$-linear maps $K_0(\delta)$ and Δ agree on the $\mathbb{Z}$-basis $\{T^n\Gamma \mid \Gamma \in \mathcal{B}, n \in \mathbb{Z}\}$ of $\mathbf{U}_T$. □

Remark 3.15. It is interesting to ask whether the properties of the comultiplication in Lemma 2.3, such as coassociativity, can be lifted to the categorical level. This question is more involved than asking whether the categorification $\mathcal{M}$ of the multiplication is a monoidal functor in Remark 2.22.

4. The linear action of $\mathbf{U}_T$ on V_n

In this section, we give a distinguished basis $\mathcal{B}_n$ of V_n and express the representation of $\mathbf{U}_T$ with respect to this basis.

4.1. **The representations V_1 and V_2.**

Definition 4.1. Let V_1 be a free $\mathbb{Z}[t^{\pm 1}]$-module with a basis $\mathcal{B}_1 = \{|0\rangle, |1\rangle\}$. The *parity* p is a $\mathbb{Z}$-grading p on V_1 given by $\mathrm{p}(|0\rangle) = 0, \mathrm{p}(|1\rangle) = 1$.

We define an action of $\mathbf{U}_T$ on V_1 by:

$$E|0\rangle = 0, \quad F|0\rangle = |1\rangle,$$
$$E|1\rangle = (1-t)|0\rangle, \quad F|1\rangle = 0,$$
$$T|0\rangle = t|0\rangle, \quad T|1\rangle = t(-1)^2|1\rangle.$$

Remark 4.2. The parities of $\mathbf{U}_T$ and V_1 are compatible with respect to the action. The operators E and F change the parity by -1 and 1, respectively:

$$\mathrm{p}(F|0\rangle) = \mathrm{p}(|1\rangle) = 1 = \mathrm{p}(F) + \mathrm{p}(|0\rangle),$$
$$\mathrm{p}(E|1\rangle) = \mathrm{p}(|0\rangle) = 0 = \mathrm{p}(E) + \mathrm{p}(|1\rangle).$$

Definition 4.3. Let $V_2 = V_1 \otimes_{\mathbb{Z}[t^{\pm 1}]} V_1$ be a free $\mathbb{Z}[t^{\pm 1}]$-module with a basis

$$\mathcal{B}'_2 = \mathcal{B}_1 \times \mathcal{B}_1 = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}.$$

The action of $\mathbf{U}_T$ on V_2 is induced by the comultiplication $\Delta : \mathbf{U}_T \to \mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$:

$$a \cdot (v \otimes w) = \Delta(a)(v \otimes w),$$

for $a \in \mathbf{U}_T, v, w \in V_1$. Note that the action of $\mathbf{U}_T \otimes_{\mathbb{Z}} \mathbf{U}_T$ on $V_1 \otimes V_1$ is the graded tensor product:

$$(a_1 \otimes a_2)(v \otimes w) = (-1)^{\mathrm{p}(a_2)\,\mathrm{p}(v)} a_1 v \otimes a_2 w.$$

Lemma 4.4. *The action of $\mathbf{U}_T$ on V_2 is given in the basis $\mathcal{B}_2$ as follows:*

$$\begin{gathered}
E|00\rangle = 0, \quad F|00\rangle = |01\rangle + t|10\rangle, \\
E|01\rangle = (1-t)|00\rangle, \quad F|01\rangle = t|11\rangle, \\
E|10\rangle = (1-t)|00\rangle, \quad F|10\rangle = -|11\rangle, \\
E|11\rangle = (1-t)|01\rangle - (1-t)|10\rangle, \quad F|11\rangle = 0, \\
T(v) = t^2 v \text{ for all } v = v_1 \otimes v_2 \in \mathcal{B}_2, \text{ where } v_1, v_2 \in \mathcal{B}_1.
\end{gathered}$$

Proof. We verify some of the formulas and leave others to the reader:

$$\begin{gathered}
T(v) = \Delta(T)(v_1 \otimes v_2) = (T \otimes T)(v_1 \otimes v_2) = T(v_1) \otimes T(v_2) = t^2 v, \\
F|00\rangle = \Delta(F)|00\rangle = (1 \otimes F + F \otimes T)|00\rangle = |01\rangle + t|10\rangle, \\
F|10\rangle = \Delta(F)|10\rangle = (1 \otimes F + F \otimes T)|10\rangle = (1 \otimes F)|10\rangle = -|11\rangle. \quad \square
\end{gathered}$$

4.2. **The representations $V_n = V_1^{\otimes n}$.** There is an action of $\mathbf{U}_T$ on the n-th tensor product $V_n = V_1^{\otimes n}$ induced by iterated comultiplication. Consider a $\mathbb{Z}[t^{\pm 1}]$-basis $\mathcal{B}'_n$ of V_n:

$$\mathcal{B}'_n = \mathcal{B}_1^{\times n} = \{\mathbf{a} = |a_1 \dots a_n\rangle \mid a_i \in \{0,1\}\}.$$

We call $\mathcal{B}'_n$ the tensor product presentation of V_n. Consider another presentation of the basis:

$$\mathcal{B}_n = \{\mathbf{x} = (x_1, \dots, x_k) \mid 1 \le x_1 < \dots < x_k \le n, 1 \le k \le n\} \sqcup \{\emptyset\}.$$

There is a one-to-one correspondence between $\mathcal{B}_n$ and $\mathcal{B}'_n$:

$$\begin{array}{ccc}
\mathcal{B}_n & \to & \mathcal{B}'_n \\
\emptyset & \mapsto & a = |0 \dots 0\rangle \\
\mathbf{x} = (x_1, \dots, x_k) & \mapsto & a = |a_1 \dots a_n\rangle
\end{array}$$

where

$$a_i = \begin{cases} 1 & \text{if } i = x_l, \text{ for some } 1 \le l \le k; \\ 0 & \text{otherwise}. \end{cases}$$

Note that there is a partition: $\mathcal{B}_n = \sqcup_{k=0}^n \mathcal{B}_{n,k}$, where $\mathcal{B}_{n,0} = \{\emptyset\}$ and

$$\mathcal{B}_{n,k} = \{\mathbf{x} = (x_1, \dots, x_k) \mid 1 \le x_1 < \dots < x_k \le n\}$$

for $1 \le k \le n$. Let $V_n = \bigoplus_{k=0}^{n} V_{n,k}$ be the corresponding decomposition of V_n, where $V_{n,k}$ is spanned by the basis $\mathcal{B}_{n,k}$ for $0 \le k \le n$.

In the $\mathbf{U}_T$-action, F converts a state from $|0\rangle$ to $|1\rangle$ for one component in the tensor product presentation $\mathcal{B}'_n$. In particular, F increases the number of $|1\rangle$ states by 1; similarly, E decreases the number of $|1\rangle$ states by 1:

$$F : V_{n,k} \to V_{n,k+1}$$
$$E : V_{n,k} \to V_{n,k-1}$$

For $\mathbf{x} = (x_1, \dots, x_k) \in \mathcal{B}_{n,k}$, let $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_{n-k}) \in \mathcal{B}_{n,n-k}$ be the increasing sequence consisting of the complement $\{1, \dots, n\} \backslash \{x_1, \dots, x_k\}$ of $\mathbf{x}$ in $\{1, \dots, n\}$. Define

$$\beta(\mathbf{x}, \bar{x}_j) = |\{l \in \{1, \dots, k\} \mid x_l < \bar{x}_j\}| + 2\, |\{l \in \{1, \dots, k\} \mid x_l > \bar{x}_j\}|\,.$$

Let $\mathbf{x} \sqcup \{\bar{x}_j\}$ be an increasing sequence obtained by adjoining $\bar{x}_j$ to $\mathbf{x}$ and $\mathbf{x}\backslash\{x_i\}$ be an increasing sequence obtained by removing x_i from $\mathbf{x}$. Now we use the basis $\mathcal{B}_n$ to describe the action.

Lemma 4.5. *The $\mathbf{U}_T$-action on V_n induced by the iterated comultiplication can be expressed as:*

$$I(\mathbf{x}) = \mathbf{x},$$
$$F(\mathbf{x}) = \sum_{j=1}^{n-k} t^{n-\bar{x}_j} (-1)^{\beta(\mathbf{x}, \bar{x}_j)} \mathbf{x} \sqcup \{\bar{x}_j\},$$
$$E(\mathbf{x}) = \sum_{i=1}^{k} \left((-1)^{1-i} \mathbf{x}\backslash\{x_i\} + \, t(-1)^{2-i} \mathbf{x}\backslash\{x_i\} \right),$$
$$T(\mathbf{x}) = t^n (-1)^{2n} \mathbf{x}.$$

for $\mathbf{x} \in \mathcal{B}_{n,k}$.

Proof. We only check the action of F:

$$\begin{aligned} F(\mathbf{x}) =& \Delta^n(F)(\mathbf{x}) \\ =& \sum_{j=1}^{n}(1\otimes\cdots 1\otimes \underset{j-\text{th}}{F}\otimes T\cdots\otimes T)(\mathbf{x}) \\ =& \sum_{j=1}^{n-k} t^{n-\bar{x}_j}(-1)^{\beta(\mathbf{x},\bar{x}_j)}\mathbf{x}\sqcup\{\bar{x}_j\}, \end{aligned}$$

where the exponent of t comes from $(n-\bar{x}_j)$'s T in the j-th term of $\Delta^n(F)$; the exponent of -1 comes from the graded tensor product and the action of T on the state $|1\rangle$: $T|1\rangle = t(-1)^2|1\rangle$. $\square$

5. The t-graded DG algebra R_n through the quiver Q_n

We define a family of t-graded DG algebras R_n through quivers Q_n for $n > 0$. The algebra R_n is closely related to the strands algebra associated to an n times punctured disk in bordered Heegaard Floer homology.

5.1. **Background on the strands algebras and the rook monoid.**

5.1.1. *The strands algebra.* We refer to [42, Section 2] for the definition of the strands algebras associated to an *arc diagram*.

Definition 5.1. An *arc diagram* $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ is a triple consisting of a collection $\mathbf{Z} = \{Z_1, \dots, Z_l\}$ of oriented line segments, a collection $\mathbf{a} = \{a_1, \dots, a_{2k}\}$ of distinct points in $\mathbf{Z}$, and a matching of $\mathbf{a}$, i.e., a 2-to-1 function $M : \mathbf{a} \to \{1, \dots, k\}$.

A surface $F(\mathcal{Z})$ can be constructed from an arc diagram $\mathcal{Z}$ by starting with a collection of rectangles $Z_j \times [0,1]$ for $j = 1, \dots, l$, and attaching a 1-handle with endpoints on $M^{-1}(i) \times \{0\}$ for each $i = 1, \dots, k$. In particular, an n times punctured disk Σ_n can be parametrized by $\mathcal{Z}(2n) = (\mathbf{Z}, \mathbf{a}, M)$, where

- $\mathbf{Z} = \{Z\}$ is a single vertical line segment;
- $\mathbf{a} = \{a_1, \dots, a_{2n}\}$ is a collection of $2n$ points in Z ordered from top to bottom; and
- the matching $M : \mathbf{a} \to \{1, \dots, n\}$ maps n pairs of adjacent points $\{a_{2i-1}, a_{2i}\}$ to $\{i\}$ for $i = 1, \dots, n$.

We fix the arc diagram $\mathcal{Z}(2n) = (\mathbf{Z}, \mathbf{a}, M)$ throughout this paper. The associated strands algebra can be described in terms of *strands diagrams*.

Definition 5.2. Given the arc diagram $\mathcal{Z}(2n)$, a *strands diagram with k strands* is a triple (S, T, ϕ), where S, T are k-element subsets of $\mathbf{a}$ and $\phi : S \to T$ is a bijection from S to T, with $\phi(i) \leq i$ for all $i \in S$.

Geometrically, a strands diagram (S, T, ϕ) with k-strands is an isotopy class of a set of k strands connecting the k points in S as a subset of the $2n$ points on the left and the k points in T as a subset of the $2n$ points on the right. The restriction that ϕ is non-increasing means that strands stay horizontal or move up when read from left to right.

The associated *strands algebra* $\mathcal{A}(2n) = \bigoplus_{k=0}^{n} \mathcal{A}(2n, k)$, where $\mathcal{A}(2n, k)$ is generated by strands diagrams with k strands with two constraints, The first constraint on a strands diagram (S, T, ϕ) is that $|S \cap \{a_{2i-1}, a_{2i}\}| \leq 1$ and $|T \cap \{a_{2i-1}, a_{2i}\}| \leq 1$ for $i = 1, \dots, n$, i.e., the number of intersection points of the strands diagram with any pair of adjacent points $\{a_{2i-1}, a_{2i}\}$ is at most 1. We call it the 1-*handle constraint*. The second constraint is that horizontal strands always appear in a pair corresponding to the matching. For instance, as shown in Figure 5, a primitive idempotent is a sum of two horizontal strands: $(S_1, T_1, \phi_1) + (S_2, T_2, \phi_2)$, where $S_1 = T_1 = \{a_5\}, S_2 = T_2 = \{a_6\}$ and ϕ_1, ϕ_2 are identities. In other words, each summand (S_1, T_1, ϕ_1) or (S_2, T_2, ϕ_2) is not a generator of $\mathcal{A}(6, 1)$. Since horizontal strands represent idempotents in the algebra, we call it the *idempotent constraint*.

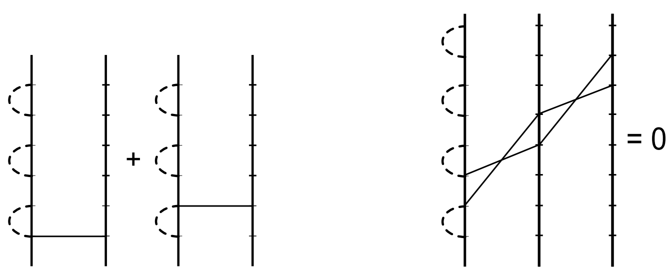
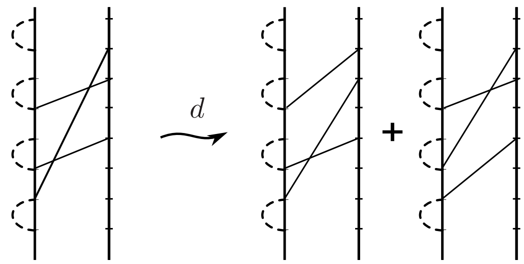


FIGURE 5. The left-hand diagram gives a primitive idempotent in $\mathcal{A}(6, 1)$; the right-hand diagram describes a double crossing.

The product $a \cdot b$ of two strands diagrams is set to be zero if the right side of a does not match the left side of b; otherwise, the product is the horizontal juxtaposition of a and b. If in the juxtaposition two strands cross each other twice, the product is set to be zero. It is called the *double crossing relation*.

The differential on the strands algebra is given by resolving a crossing in a strand diagram. More precisely, the boundary of a strand diagram is the sum over all ways of resolving one crossing of

the diagram, where those diagrams with a double crossing are removed from the sum. An example is given below.



5.1.2. *The rook monoid.* Since the matching in the arc diagram $\mathcal{Z}(2n)$ just identifies a_{2i-1} and a_{2i} in each pair, we use *rook diagrams* to describe the strands algebra $\mathcal{A}(2n)$. The *rook monoid* was defined in [35].

Definition 5.3. Let n be a positive integer and $\mathbf{n} = \{1, \dots, n\}$. The *rook monoid* $\mathcal{R}_n$ is the set of all one-to-one maps σ with domain $I(\sigma) \subset \mathbf{n}$ and range $J(\sigma) \subset \mathbf{n}$. The multiplication on $\mathcal{R}_n$ is given by composition of maps.

Here we use a diagrammatic presentation of the rook monoid, called *rook diagrams*, given in [6]. A rook diagram associated to an element $\sigma \in \mathcal{R}_n$ is a graph on two rows of n vertices such that vertex i in the bottom row is connected to vertex j in the top row if and only if $\sigma(i) = j$. The multiplication is given by vertical concatenation of two rook diagrams.

5.1.3. *From strands diagrams to rook diagrams.* We describe the translation from the strands diagrams in $\mathcal{A}(2n)$ to the rook diagrams in $\mathcal{R}_n$ as follows. We first rotate a strand diagram counterclockwise by $\frac{\pi}{2}$. Then we replace the identified points $\{a_{2i-1}, a_{2i}\}$ of i-th pair in a strand diagram by the i-th vertex in a rook diagram. Therefore, the n pairs of $2n$ points are replaced by a row of n vertices. In a strand diagram (S, T, ϕ), there is at most one strand connecting to each pair of identified points from the 1-handle constraint on strand diagrams. Hence there is at most one strand connecting to any vertex in the corresponding rook diagram. We label the i-th vertex in the bottom row by the state $|1\rangle$ if $|S \cap \{a_{2i-1}, a_{2i}\}| = 1$; otherwise, we label it by the state $|0\rangle$. Similarly, we label the i-th vertex in the top row by the state $|1\rangle$ if $|T \cap \{a_{2i-1}, a_{2i}\}| = 1$; otherwise, we label it by the state $|0\rangle$. Finally, we get a rook diagram with decorations in the states $\{|0\rangle, |1\rangle\}$.

As shown in Figure 6, the idempotent as a sum of two horizontal strands in the strands algebra is translated to a single vertical rook diagram id; an upward strand connecting two points in a pair in

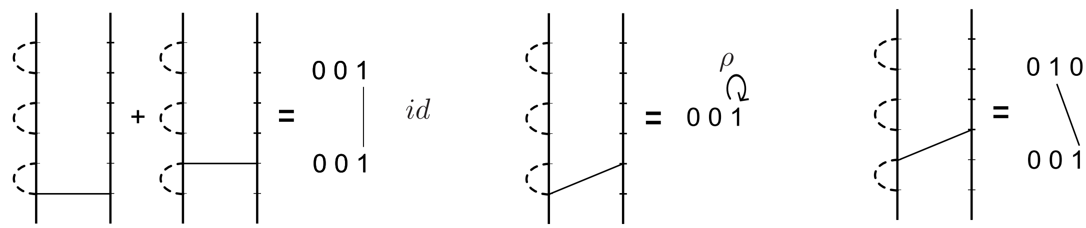


FIGURE 6. The correspondence for $\mathcal{A}(6,1)$: the left one is the idempotent id; the middle one is the nilpotent element ρ; the right one is a non-vertical strand.

the strands algebra is translated to a loop ρ attached at the corresponding state $|1\rangle$. Note that the loop ρ is nilpotent in the strands algebra: $\rho^2 = 0$. Correspondingly, the square of any loop is zero in the generalized rook algebra. Since the strands diagrams stay horizontal or move up, the corresponding rook diagrams always have negative or infinity slopes, i.e., they stay vertical or move to the left when read from bottom to top. Finally, we call these generalized rook diagrams possibly with loops as *left-veering decorated rook diagrams*. We will use *decorated rook diagrams* for simplicity.

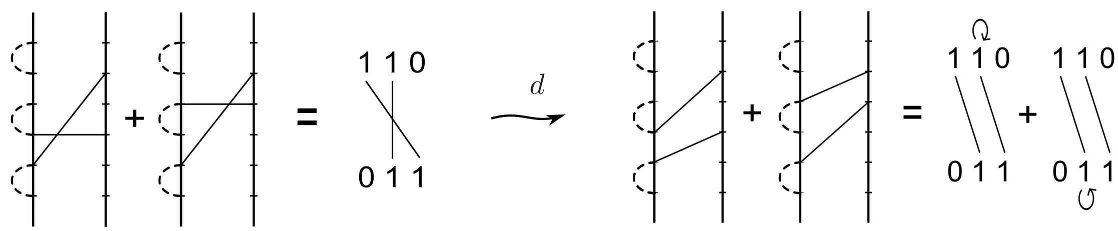


FIGURE 7. The differential of a crossing: $d(r((2,3) \xrightarrow{1,1,(0,0)} (1,2)))$.

Since a vertical strand in a decorated rook diagram corresponds to a sum of two terms in the strands algebra, the resolution of such a crossing in a decorated rook diagram contains two terms with loops as shown in Figure 7.

5.1.4. *New ingredients.* We first add a new type of decorated rook diagrams and a new differential to deform a relation in the strands algebra $\mathcal{A}(2n)$:

$$(S,T,\phi)\cdot(S',T',\phi') = 0,$$

if $T \cap \{a_{2i-1}, a_{2i}\} = \{a_{2i-1}\}$ and $S' \cap \{a_{2i-1}, a_{2i}\} = \{a_{2i}\}$ for some i. See Figure 8. This new decorated rook diagram is given by adding a marking on the diagram at the position corresponding to the pair $\{a_{2i-1}, a_{2i}\}$. In general, there exist decorated rook diagrams with crossings and markings.

The resolution of such a decorated rook diagram is a combination of resolutions of crossings and those of markings. See Figure 9.

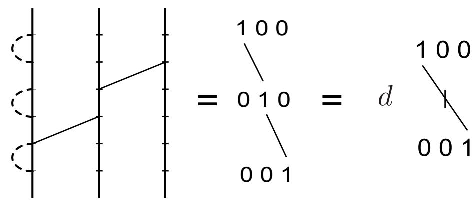


FIGURE 8. The differential of a marking: $d(r((3) \xrightarrow{1,0,(1)} (1)))$.

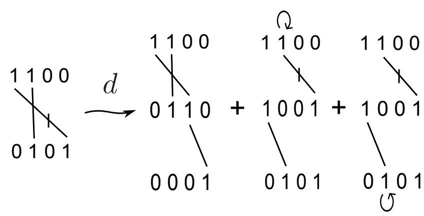


FIGURE 9.

We introduce the notion of *elementary decorated rook diagrams* which cannot be decomposed as a concatenation of two nontrivial pieces. The elementary decorated rook diagrams will give generators in the algebra R_n later. Notice that n-tuples of $\{|0\rangle, |1\rangle\}$ are elements in the basis $\mathcal{B}_n$ of the representation V_n. Hence, a decorated rook diagram can be viewed as a map from one element of $\mathcal{B}_n$ in the bottom row to the other element of $\mathcal{B}_n$ in the top row.

Definition 5.4. An *elementary decorated rook diagram* $\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}$ is a decorated rook diagram with s_1 crossings and $s_0(\mathbf{v})$ markings corresponding to

$$\sigma : \mathbf{x} = (x_1, \ldots, x_k) \to \mathbf{y} = (y_1, \ldots, y_k)$$

such that there exist $i \in \{1, \ldots, k\},\ s_1 \geq 0,\ \mathbf{v} \in \mathbb{N}^{s_1+1}$ and

$$\sigma(x_{i+s_1}) = y_i,$$
$$\sigma(x_j) = y_{j+1} = x_j \text{ for } j \in \{i, \ldots, i+s_1-1\},$$
$$\sigma(x_j) = y_j = x_j \text{ for } j \notin \{i, i+1, \ldots, i+s_1\},$$
$$\mathbf{v} = (x_i - y_i - 1, \ldots, x_{i+s_1} - y_{i+s_1} - 1) \in \mathbb{N}^{s_1+1};$$

where $s_0(\mathbf{v}) = \sum_{l=0}^{s_1} v_l$.

The algebraic definition above is technical while geometric diagrams are easier to follow as shown in Figure 10. Given an elementary decorated rook diagram $\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}$, $\mathbf{x}$ and $\mathbf{y}$ only differ at two positions x_{i+s_1}, y_i which are connected by a non-vertical strand. On the strand, there are s_1 crossings with s_1 vertical strands and $s_0(\mathbf{v})$ markings. In other words, all possible crossings and markings must be on the strand. The vector $\mathbf{v}$ counts the numbers of $|0\rangle$ states between the j-th $|1\rangle$ states $\{x_j\}$ in $\mathbf{x}$ and $\{y_j\}$ in $\mathbf{y}$ for $j = i, \ldots, i+s_1$.

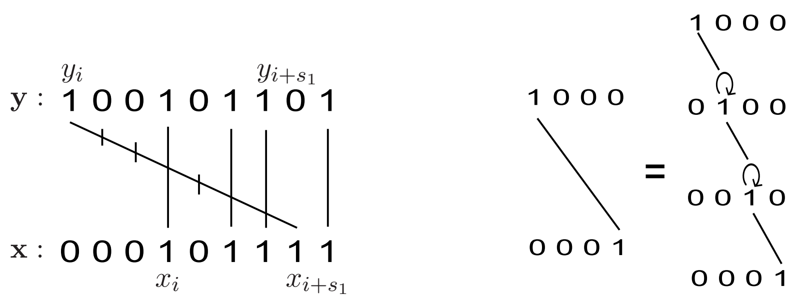
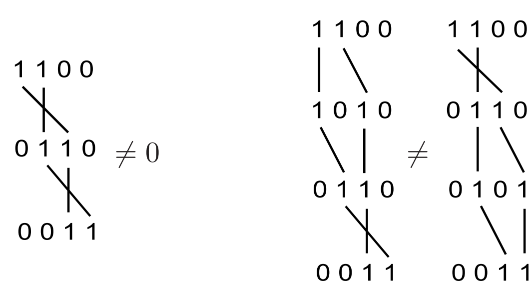


FIGURE 10. The left-hand diagram is an elementary rook diagram $\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}$, where $i = 1, s_1 = 3, \mathbf{v} = (2, 1, 0, 0)$; in the right-hand diagram, we use the left-hand side to denote the composition of elementary diagrams on the right-hand side which represents a nontrivial element in the cohomology of the algebra R_n.

Relations for concatenation of decorated rook diagrams are quite different from those for the strands diagrams:

- The double crossing in decorated rook diagrams is not zero.
- An isotopy of a crossing does not give the same decorated rook diagram.

For more detail about the relations, refer to the definition of the algebra R_n in Section 5.3.



5.1.5. *The algebraic description of the resolution.* For any elementary decorated rook diagram $\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}$, its resolution is a sum of $2s_1 + s_0(\mathbf{v})$ terms obtained by resolving each crossing and marking. Although the resolution is easy to understand from diagrams, the algebraic description in the following is technical. For $0 \leq s \leq s_1 - 1$, consider a decomposition of $\mathbf{v} = (v_0, ..., v_{s_1})$:

$$\mathbf{v}^1(s) = (v_0, ..., v_s), \quad \overline{\mathbf{v}^1(s)} = (v_{s+1}, ..., v_{s_1}),$$

according to resolving the $(s+1)$st crossing. There exist an intermediate state $\mathbf{z}^s$ between $\mathbf{x}$ and $\mathbf{y}$ and two elementary decorated rook diagrams:

$$\mathbf{x} \xrightarrow{i,s,\mathbf{v}^1(s)} \mathbf{z}^s, \quad \mathbf{z}^s \xrightarrow{i+s+1,s_1-s-1,\overline{\mathbf{v}^1(s)}} \mathbf{y}.$$

Then the resolution of such a crossing is a sum of two terms each of which is a product of these elementary decorated rook diagrams together with a loop.

For $1 \leq t \leq s_0(\mathbf{v})$, let $f(t) \in \{0, ..., s_1\}$ such that $\sum\limits_{l=0}^{f(t)-1} v_l < t \leq \sum\limits_{l=0}^{f(t)} v_l$. Consider another decomposition of $\mathbf{v} = (v_0, ..., v_{s_1})$:

$$\mathbf{v}^0(t) = (v_0, ..., v_{f(t)-1}, t - \sum_{l=0}^{f(t)-1} v_l - 1), \quad \overline{\mathbf{v}^0(t)} = (\sum_{l=0}^{f(t)} v_l - t, v_{f(t)+1}, ..., v_{s_1}),$$

according to resolving the t-th marking. There exist an intermediate state $\mathbf{w}^t$ between $\mathbf{x}$ and $\mathbf{y}$ and two elementary decorated rook diagrams:

$$\mathbf{x} \xrightarrow{i+f(t),s_1-f(t),\overline{\mathbf{v}^0(t)}} \mathbf{w}^t, \quad \mathbf{w}^t \xrightarrow{i,f(t),\mathbf{v}^0(t)} \mathbf{y}.$$

Then the resolution of such a marking is a product of these elementary decorated rook diagrams. We will use these resolutions to define a differential in the next section.

5.2. **The quiver Q_n.** In this section, we construct the quiver $Q_n = \sqcup_{k=0}^{n} Q_{n,k}$ for $n > 0$.

Definition 5.5 (Quiver $Q_{n,k} = (V(Q_{n,k}), A(Q_{n,k}))$)**.**

(1) Let $V(Q_{n,k}) = \mathcal{B}_{n,k}$ be the set of vertices.
(2) Let $A(Q_{n,k})$ be the set of arrows consisting of two types:

$$\text{Loops: } \{\mathbf{x} \xrightarrow{i} \mathbf{x} \mid i = 1, \ldots, k;\ \mathbf{x} \in \mathcal{B}_{n,k}\},$$

$$\text{Arrows: } \{\text{elementary decorated rook diagrams } \mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}\}$$

Example 5.6 (Quiver $\Gamma_{4,2}$). $V(\Gamma_{4,2}) = \{(3,4),(2,4),(1,4),(2,3),(1,3),(1,2)\}$. For $\mathbf{x} = (x_1, x_2)$, there exist two loops, one for each x_i. There are 6 arrows without crossings or markings,

$$\{(3,4) \xrightarrow{1,0,(0)} (2,4),\ (2,4) \xrightarrow{1,0,(0)} (1,4),\ (2,4) \xrightarrow{2,0,(0)} (2,3),$$
$$(1,4) \xrightarrow{2,0,(0)} (1,3),\ (2,3) \xrightarrow{1,0,(0)} (1,3),\ (1,3) \xrightarrow{2,0,(0)} (1,2)\}$$

and 6 arrows with crossings or markings:

$$\{(3,4) \xrightarrow{1,0,(1)} (1,4),\ (3,4) \xrightarrow{1,1,(0,0)} (2,3),\ (3,4) \xrightarrow{1,1,(1,0)} (1,3),$$
$$(1,4) \xrightarrow{2,0,(1)} (1,2),\ (2,3) \xrightarrow{1,1,(0,0)} (1,2),\ (2,4) \xrightarrow{1,1,(0,1)} (1,2)\}$$

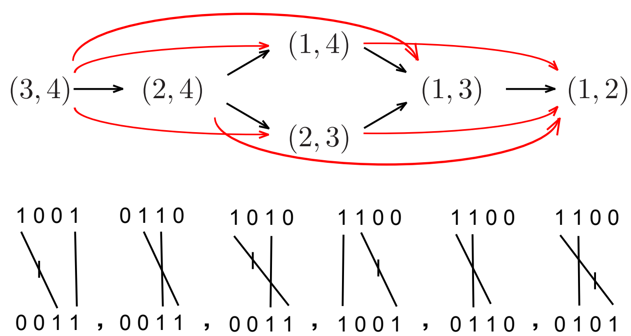


FIGURE 11. The top diagram describes the quiver $\Gamma_{4,2}$, where black lines denote arrows without crossings or markings and red lines denote arrows with crossings or markings which are represented in the bottom diagram.

5.3. **The t-graded DG algebra R_n.** We define the t-graded DG algebra $R_n = \bigoplus_{k=0}^{n} R_{n,k}$, where $R_{n,k} = \mathbb{F}_2 Q_{n,k}/\sim$ is a quotient of the path algebra $\mathbb{F}_2 Q_{n,k}$ of the quiver $Q_{n,k}$ with a differential. $\mathbb{F}_2$ is fixed as the ground field throughout the paper.

Definition 5.7 (t-graded DG algebra R_n)**.** R_n is an associative t-graded $\mathbb{F}_2$-algebra with a differential d and a grading $\deg = (\deg_h, \deg_t) \in \mathbb{Z}^2$.

(A) R_n has idempotents $e(\mathbf{x})$ for each vertex $\mathbf{x}$ in Q_n, generators $\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})$ for each loop $\mathbf{x} \xrightarrow{i} \mathbf{x}$ and $r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ for each arrow $\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}$ in Q_n. The relations consist of 4 groups:

(i) idempotents:

$$e(\mathbf{x}) \cdot e(\mathbf{y}) = \delta_{\mathbf{x},\mathbf{y}} \cdot e(\mathbf{x}) \text{ for all } \mathbf{x}, \mathbf{y},$$
$$e(\mathbf{x}) \cdot \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) = \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \cdot e(\mathbf{x}) = \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \text{ for all } \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}),$$
$$e(\mathbf{x}) \cdot r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}) = r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}) \cdot e(\mathbf{y}) = r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}) \text{ for all } r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y});$$

(ii) nilpotent elements:

$$\rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \cdot \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) = 0 \text{ for all } \rho(\mathbf{x} \xrightarrow{i} \mathbf{x});$$

(iii) commutativity:

$$\rho(\mathbf{x} \xrightarrow{i'} \mathbf{x}) \cdot \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) = \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \cdot \rho(\mathbf{x} \xrightarrow{i'} \mathbf{x}) \text{ if } i' \neq i,$$
$$\rho(\mathbf{x} \xrightarrow{i'} \mathbf{x}) \cdot r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}) = r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}) \cdot \rho(\mathbf{y} \xrightarrow{i'} \mathbf{y}) \text{ if } i' \notin \{i, \ldots, i+s_1\},$$

$$r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}) \cdot r(\mathbf{y} \xrightarrow{i',s_1',\mathbf{v}'} \mathbf{z}) = r(\mathbf{x} \xrightarrow{i',s_1',\mathbf{v}'} \mathbf{w}) \cdot r(\mathbf{w} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{z}) \text{ if } x_{i+s_1} < z_{i'};$$

(iv) sliding over a crossing:

$$\rho(\mathbf{x} \xrightarrow{i'} \mathbf{x}) \cdot r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}) = r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}) \cdot \rho(\mathbf{y} \xrightarrow{i'+1} \mathbf{y}) \text{ if } i' \in \{i, \ldots, i+s_1-1\}, s_1 > 0.$$

(B) The differential is defined on generators by:

$$\begin{aligned} d(r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})) = & \sum_{s=0}^{s_1-1} (r(\mathbf{x} \xrightarrow{i,s,\mathbf{v}^1(s)} \mathbf{z}^s) \cdot r(\mathbf{z}^s \xrightarrow{i+s+1,s_1-s-1,\overline{\mathbf{v}^1(s)}} \mathbf{y}) \cdot \rho(\mathbf{y} \xrightarrow{i+s+1} \mathbf{y}) \\ & + \rho(\mathbf{x} \xrightarrow{i+s} \mathbf{x}) \cdot r(\mathbf{x} \xrightarrow{i,s,\mathbf{v}^1(s)} \mathbf{z}^s) \cdot r(\mathbf{z}^s \xrightarrow{i+s+1,s_1-s-1,\overline{\mathbf{v}^1(s)}} \mathbf{y})) \\ & + \sum_{t=1}^{s_0} r(\mathbf{x} \xrightarrow{i+f(t),s_1-f(t),\overline{\mathbf{v}^0(t)}} \mathbf{w}^t) \cdot r(\mathbf{w}^t \xrightarrow{i,f(t),\mathbf{v}^0(t)} \mathbf{y}) \end{aligned}$$

for $s_0(\mathbf{v}) + s_1 > 0$; $d(r) = 0$ otherwise. It is extended by the Leibniz rule

$$d(r_1 \cdot r_2) = d(r_1) \cdot r_2 + r_1 \cdot d(r_2)$$

for $r_1, r_2 \in R_n$.

(C) The grading $\deg = (\deg_h, \deg_t)$ is defined on generators by:

$$\deg(e(\mathbf{x})) = (0, 0),$$
$$\deg(\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})) = (-1, -1),$$
$$\deg(r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})) = (1 - s_1, 1 + s_0).$$

Remark 5.8. Geometrically, the commutativity relations (Relation (iii)) correspond to isotopies of stackings of disjoint rook diagrams.

Remark 5.9. The differential d corresponds to resolutions of crossings and markings. R_n can be viewed as a double complex: d admits a decomposition $d = d_0 + d_1$ such that

$$d_0^2 = d_1^2 = 0, \quad d_0 d_1 = d_1 d_0,$$

where d_0 and d_1 correspond to resolutions of markings and crossings, respectively. This will become useful in Lemma 5.11.

Lemma 5.10. *d is well-defined and is a differential on R_n.*

Proof. We use the geometric description of d in terms of resolving crossings and markings.

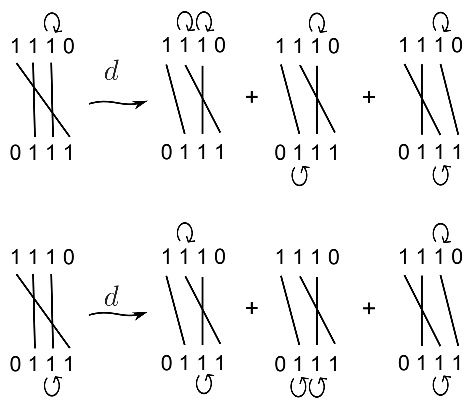


FIGURE 12. Differentials of both sides in the sliding relation.

(1) Well-definition. We show that d is well-defined under the relations of R_n. Since the commutativity relations correspond to isotopies of stackings of disjoint rook diagrams, their resolutions commute as well. The pictorial proof of the invariance of the differential under the sliding relation (Relation (iv)) is given in Figure 12. Resolutions of both sides are obviously the same under the sliding relation except for the resolution of the crossing over which the loop slides.

(2) Verification that d is a differential. It is easy to check that d is of degree $(1,0)$. We have to show that $d^2(r) = 0$ for any generator $r \in R_n$. In the expansion of $d^2(r)$, any term comes from a resolution of two of crossings and markings. Then the coefficient of each term is even since there are two ways to resolve them depending on the different orders of resolutions. Hence, $d^2 = 0$ since we are working in $\mathbb{F}_2$. $\square$

Lemma 5.11. *The cohomology $H(R_n)$ is generated by idempotents $e(\mathbf{x})$, loops $\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})$ and arrows $r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ without crossings or markings, i.e., $s_1 = s_0(\mathbf{v}) = 0$.*

Proof. It is easy to see that $R_n = \bigoplus_{\mathbf{x},\mathbf{y} \in \mathcal{B}_n} R_n(\mathbf{x},\mathbf{y})$, where $R_n(\mathbf{x},\mathbf{y})$ is the subspace of R_n generated by all the arrows from $\mathbf{x}$ to $\mathbf{y}$. It suffices to prove the lemma for $R_n(\mathbf{x},\mathbf{y})$. Since the differential $d = d_0 + d_1$ can be decomposed into two differentials, we have a double complex

$$C = \bigoplus_{p,q} R_n(\mathbf{x},\mathbf{y})_{p,q},$$

where $R_n(\mathbf{x},\mathbf{y})_{p,q}$ is the subspace of $R_n(\mathbf{x},\mathbf{y})$ generated by all $r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ with $s_1 = p, s_0 = q$, and the horizontal and vertical differentials are d_1 and d_0, respectively.

$$\begin{tikzcd}
R_n(\mathbf{x},\mathbf{y})_{0,2} \arrow[d, "d_0"] & R_n(\mathbf{x},\mathbf{y})_{1,2} \arrow[l, "d_1"'] \arrow[d, "d_0"] & \cdots \arrow[l, "d_1"'] \arrow[d, "d_0"] \\
R_n(\mathbf{x},\mathbf{y})_{0,1} \arrow[d, "d_0"] & R_n(\mathbf{x},\mathbf{y})_{1,1} \arrow[l, "d_1"'] \arrow[d, "d_0"] & \cdots \arrow[l, "d_1"'] \arrow[d, "d_0"] \\
R_n(\mathbf{x},\mathbf{y})_{0,0} & R_n(\mathbf{x},\mathbf{y})_{1,0} \arrow[l, "d_1"'] & \cdots \arrow[l, "d_1"']
\end{tikzcd}$$

Then the differential in the total complex $Tot(C)$ is $d = d_0 + d_1$ in R_n. Note that the double complex C is finite since there are at most n crossings and markings in any arrow. Consider the two spectral sequences of C from the two filtrations which converge to the homology of $Tot(C)$ [40, Section 5.6]. Let $E^1_{p,q} = H^v_q(C_{p,*})$ and $'E^1_{p,q} = H^h_p(C_{*,q})$ be the first pages by taking the homology of the vertical differential d_0 and the horizontal differential d_1, respectively. We will show that

$$E^1_{p,q} = 0 \text{ for } q > 0;$$

$$'E^1_{p,q} = 0 \text{ for } p > 0.$$

Therefore, $H_{p+q}Tot(C) = 0$ for $p + q > 0$, i.e., $H(R_n)$ is generated by loops and arrows without crossings or markings.

For E^1_{pq} with $q > 0$, suppose $d_0(r) = 0$, where $r = \sum_i r_i \in C_{p,q}$. Assume further that r is primitive, i.e., any nontrivial partial sum of $\sum_i r_i$ is non-closed. Each r_i can be represented by a product of rook diagrams which give a collection of intermediate states between $\mathbf{x}$ and $\mathbf{y}$. Moreover, the decomposition is uniquely determined up to the commutativity and sliding relations in Definition 5.7. The key observation is that the resolution of a marking in a decorated rook diagram only locally changes the marking α to the gap β, as shown in Figure 13. In other words, there exists a common collection of intermediate states for r_i and all terms in the expansion $d_0(r_i)$. On the other hand, the coefficient of each term in $d_0(r_i)$ is even since $d_0(r) = 0$. Hence a common collection of intermediate states exists for r_i and r_j if $d_0(r_i)$ and $d_0(r_j)$ have some terms in common. Since we assume r is primitive, there exists a common collection of intermediate states $\{\mathbf{z}^s\}$ for all r_i, i.e., each r_i can be decomposed into a path from $\mathbf{x}$ to $\mathbf{y}$ through $\{\mathbf{z}^s\}$ in Q_n:

$$\mathbf{x} \to \mathbf{z}^1 \to \cdots \to \mathbf{z}^s \to \mathbf{y}.$$

Moreover, there exists a concatenation Γ of decorated rook diagrams from $\mathbf{x}$ to $\mathbf{y}$ and a finite set $\mathcal{I}$ indexed by all markings in Γ such that each r_i is represented by Γ_i which is obtained from Γ by changing some of the markings to gaps.

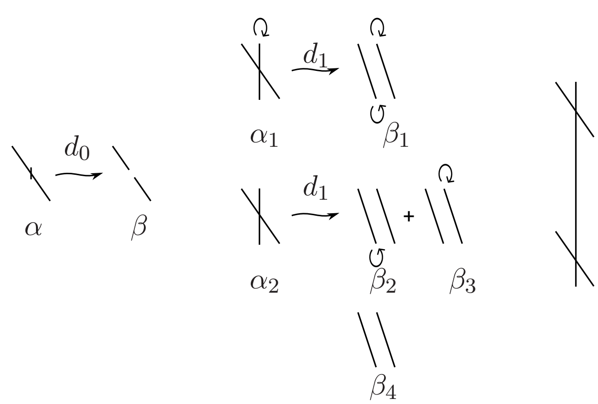


FIGURE 13. The left picture is W_0; the middle is W_1; the right is $W_1 \otimes_S W_1$.

We construct a *chain complex of markings* $C_0(r)$ for $r \in R_n$ such that $d_0(r) = 0$ as follows. Let

$$W_0 = \langle \alpha \rangle \xrightarrow{d_0} \langle \beta \rangle$$

be a two-dimensional vector space generated by a marking α and a gap β; the differential d_0 resolves a marking α and yields a gap β. The homology of (W_0, d_0) is zero. Note that (W_0, d_0) is the local model for one marking in Γ. Define the chain complex of markings $C_0(r)$ by $|\mathcal{I}|$-th tensor

product $W_0^{\otimes|\mathcal{I}|}$ of W_0 over $\mathbb{F}_2$. In other words, $C_0(r)$ encodes the information of all markings in Γ. Then each $r_i \in R_n$ corresponds to a generator in the chain complex and the differential d_0 in R_n corresponds to the differential in $W_0^{\otimes|\mathcal{I}|}$. We compute the homology of $W_0^{\otimes|\mathcal{I}|}$ in the following.

Recall the Künneth formula from [40, Theorem 3.6.3]: If P and Q are right and left complexes of R-modules such that P_n and $d(P_n)$ are flat for each n, then there is an exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes H_p(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{p+q=n-1} Tor_1^R(H_p(P), H_p(Q)) \to 0.$$

The homology of $(W_0^{\otimes|\mathcal{I}|}, d_0)$ is zero by taking $R = \mathbb{F}_2$ and $P = Q = W_0$. It is easy to see that $r = \sum_i r_i$ corresponds a closed element in $W^{\otimes|\mathcal{I}|}$. Then there exists another element $w \in W^{\otimes|\mathcal{I}|}$ such that $d_0(w) = r$ since the homology of $(W_0^{\otimes|\mathcal{I}|}, d_0)$ is zero. Hence, there exists a corresponding element w in R_n such that $d_0(w) = r \in R_n$.

For ${}'E^1_{pq}$, the proof is similar to that for E^1_{pq}. A key difference is that the collection of local diagrams consists of 6 patterns as shown in Figure 13. Let (W_1, d_1) be the chain complex given by locally resolving a crossing:

$$\begin{array}{ccc} \langle \alpha_1, \alpha_2 \rangle & \xrightarrow{d_1} & \langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle \\ \alpha_1 & \mapsto & \beta_1, \\ \alpha_2 & \mapsto & \beta_2 + \beta_3. \end{array}$$

Then (W_1, d_1) is the local model for one crossing. Notice that (W_1, d_1) can be viewed as a chain complex of S-bimodules, where $S = \langle 1, \rho \mid \rho^2 = 0 \rangle$ acts on W_1 by adding a loop ρ to the decorated rook diagrams.

Similarly, we construct a *chain complex of crossings* $C_1(r')$ as a tensor product of W_1's for any element $r' \in R_n$ such that $d_1(r') = 0$. The chain complex $C_1(r')$ is supposed to encode the information of all crossings in Γ' associated to r'. Since the loop ρ could slide over a crossing and along a vertical strand, the tensor product of two W_1's is over S if the corresponding two crossings can be connected by a vertical strand as shown in Figure 13; otherwise, the tensor product is over $\mathbb{F}_2$.

It is easy to verify the conditions for $W_1 \otimes_S W_1$ and $W_1 \otimes_{\mathbb{F}_2} W_1$ in the Künneth formula. The homology $H_1(W_1)$ is zero at degree 1 and $H_0(W_1)$ is isomorphic to the ring S. Hence, $H_0(W_1)$ is free as left and right S modules and the Tor group in the Künneth formula vanishes. In both cases, we have

$$H_n(W_1 \otimes_R W_1) \cong \bigoplus_{p+q=n} H_p(W_1) \otimes_R H_p(W_1).$$

It follows that the homology of $C_1(r')$ is zero at degree greater than 0. Hence, $'E^1_{pq} = 0$ for $p > 0$ and we conclude the proof. □

Remark 5.12. The first page E^1_{pq} given by the differential d_0 for resolving markings is very close to the strands algebra $\mathcal{A}(2n)$. They only differ at the relation of a double crossing which is set to be zero in the strands algebra.

Let $r(\mathbf{x} \xrightarrow{i} \mathbf{y})$ denote the class $[r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})]$ with $s_1 = s_0(\mathbf{v}) = 0$ in the cohomology $H(R_n)$.

Proposition 5.13. *The cohomology $H(R_n)$ is an associative t-graded DG algebra with a trivial differential. It has idempotents $e(\mathbf{x})$ for each vertex $\mathbf{x}$ in Q_n, generators $\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})$ for each loop $\mathbf{x} \xrightarrow{i} \mathbf{x}$ and $r(\mathbf{x} \xrightarrow{i} \mathbf{y})$ for each arrow $\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}$ with $s_1 + s_0(\mathbf{v}) = 0$ in Q_n. The relations consist of 4 groups:*
(i) idempotents:

$$e(\mathbf{x}) \cdot e(\mathbf{y}) = \delta_{\mathbf{x},\mathbf{y}} \cdot e(\mathbf{x}) \textit{ for all } \mathbf{x}, \mathbf{y},$$
$$e(\mathbf{x}) \cdot \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) = \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \cdot e(\mathbf{x}) = \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \textit{ for all } \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}),$$
$$e(\mathbf{x}) \cdot r(\mathbf{x} \xrightarrow{i} \mathbf{y}) = r(\mathbf{x} \xrightarrow{i} \mathbf{y}) \cdot e(\mathbf{y}) = r(\mathbf{x} \xrightarrow{i} \mathbf{y}) \textit{ for all } r(\mathbf{x} \xrightarrow{i} \mathbf{y});$$

(ii) unstackability relations (R1):

$$\rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \cdot \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) = 0,$$
$$r(\mathbf{x} \xrightarrow{i} \mathbf{y}) \cdot r(\mathbf{y} \xrightarrow{i} \mathbf{z}) = 0;$$

(iii) commutativity relations (R2):

$$\rho(\mathbf{x} \xrightarrow{i'} \mathbf{x}) \cdot \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) = \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \cdot \rho(\mathbf{x} \xrightarrow{i'} \mathbf{x}) \textit{ if } i' \neq i,$$
$$\rho(\mathbf{x} \xrightarrow{i'} \mathbf{x}) \cdot r(\mathbf{x} \xrightarrow{i} \mathbf{y}) = r(\mathbf{x} \xrightarrow{i} \mathbf{y}) \cdot \rho(\mathbf{y} \xrightarrow{i'} \mathbf{y}) \textit{ if } i' \neq i,$$
$$r(\mathbf{x} \xrightarrow{i} \mathbf{y}) \cdot r(\mathbf{y} \xrightarrow{i'} \mathbf{z}) = r(\mathbf{x} \xrightarrow{i'} \mathbf{w}) \cdot r(\mathbf{w} \xrightarrow{i} \mathbf{z}) \textit{ if } x_i < z_{i'};$$

(iv) relation (R3) from the differential of a crossing:

$$\rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \cdot r(\mathbf{x} \xrightarrow{i} \mathbf{y}) \cdot r(\mathbf{y} \xrightarrow{i+1} \mathbf{z}) = r(\mathbf{x} \xrightarrow{i} \mathbf{y}) \cdot r(\mathbf{y} \xrightarrow{i+1} \mathbf{z}) \cdot \rho(\mathbf{z} \xrightarrow{i+1} \mathbf{z}) \textit{ if } z_{i+1} = x_i.$$

The t-graded DG algebra R_n is formal since its cohomology $H(R_n)$ is concentrated along the line $\deg_h - \deg_t = 0$ by Lemma 5.11.

Lemma 5.14. *The t-graded DG algebra R_n is quasi-isomorphic to its cohomology $H(R_n)$.*

Consider a collection of projective DG R_n-modules: $\{P(\mathbf{x}) = R_n \cdot e(\mathbf{x}) \mid \mathbf{x} \in \mathcal{B}_n\}$, and a collection of projective DG $H(R_n)$-modules: $\{PH(\mathbf{x}) = H(R_n) \cdot e(\mathbf{x}) \mid \mathbf{x} \in \mathcal{B}_n\}$.

Definition 5.15. (1) Let $DGP(R_n)$ be the smallest full subcategory of $DG(R_n)$ which contains the projective DG R_n-modules $\{P(\mathbf{x}) \mid \mathbf{x} \in \mathcal{B}_n\}$ and is closed under the cohomological grading shift functor $[1]$, the t-grading shift functor $\{1\}$ and taking mapping cones.
(2) Let $DGP(H(R_n))$ be the smallest full subcategory of $DG(H(R_n))$ which contains the projective DG R_n-modules $\{PH(\mathbf{x}) \mid \mathbf{x} \in \mathcal{B}_n\}$ and is closed under the cohomological grading shift functor $[1]$, the t-grading shift functor $\{1\}$ and taking mapping cones.

Since R_n is formal, the triangulated category $H^0(DG(R_n))$ is equivalent to $H^0(DG(H(R_n)))$. Furthermore, it is easy to see the following equivalence of their subcategories.

Lemma 5.16. *The triangulated categories $H^0(DGP(R_n))$ and $H^0(DGP(H(R_n)))$ are equivalent. Hence there are isomorphisms of $\mathbb{Z}[t^{\pm 1}]$-modules:*

$$K_0(H^0(DGP(R_n))) \cong K_0(H^0(DGP(H(R_n)))) \cong \mathbb{Z}[t^{\pm 1}]\langle \mathcal{B}_n \rangle \cong V_n.$$

5.4. **The t-graded DG algebra $A \boxtimes R_n$.** In this section, we define a DG algebra $A \boxtimes R_n$ by adding an extra differential on the tensor product $A \otimes_{\mathbb{F}_2} R_n$. The extra differential will only be used in the construction of the DG $(H(R_n), A \boxtimes R_n)$-bimodule C_n in Section 6.4. The definition of $A \boxtimes R_n$ is rather technical and the reader can pretend it is $A \otimes R_n$ at a first reading.

Definition 5.17. $A \boxtimes R_n$ is an associative t-graded DG $\mathbb{F}_2$-algebra with a differential d and a grading $\deg = (\deg_h, \deg_t) \in \mathbb{Z}^2$.

(A) $A \boxtimes R_n$ has generators

$$e(\Gamma) \boxtimes r, \quad a \boxtimes e(\mathbf{x}),$$

for $\Gamma \in \mathcal{B}, r \in R_n, a \in A, \mathbf{x} \in \mathcal{B}_n$;

$$\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}),$$

for $1 \leq i \leq k < n$ and $\mathbf{x} \in \mathcal{B}_{n,k}$ such that $x_i = n - k + i$;

$$\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}),$$

for $1 \leq j \leq k < n$ and $\mathbf{x} \in \mathcal{B}_{n,k}$ such that $x_j = j$.

(B) The differential is defined on generators in the following and extended by the Leibniz rule.

$$
\begin{aligned}
&(6) & d(a \boxtimes e(\mathbf{x})) &= 0;\\
&(7) & d(e(\Gamma) \boxtimes r) &= e(\Gamma) \boxtimes d(r);\\
&(8) & d(\rho(I\mathbf{x} \xrightarrow{k} EF\mathbf{x})) &= (\rho(I,EF) \boxtimes e(\mathbf{x})) \cdot (e(EF) \boxtimes \rho(\mathbf{x} \xrightarrow{k} \mathbf{x}))\\
&&&\quad +(e(I) \boxtimes \rho(\mathbf{x} \xrightarrow{k} \mathbf{x})) \cdot (\rho(I,EF) \boxtimes e(\mathbf{x}));\\
&(9) & d(\rho(EF\mathbf{x} \xrightarrow{1} I\mathbf{x})) &= (\rho(EF,I) \boxtimes e(\mathbf{x})) \cdot (e(I) \boxtimes \rho(\mathbf{x} \xrightarrow{1} \mathbf{x}))\\
&&&\quad +(e(EF) \boxtimes \rho(\mathbf{x} \xrightarrow{1} \mathbf{x})) \cdot (\rho(EF,I) \boxtimes e(\mathbf{x}));\\
&(10) & d(\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x})) &= \rho(I\mathbf{x} \xrightarrow{i+1} EF\mathbf{x}) \cdot (e(EF) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}))\\
&&&\quad +(e(I) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x})) \cdot (\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x})) \text{ for } i<k;\\
&(11) & d(\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})) &= \rho(EF\mathbf{x} \xrightarrow{j-1} I\mathbf{x}) \cdot (e(I) \boxtimes \rho(\mathbf{x} \xrightarrow{j} \mathbf{x}))\\
&&&\quad +(e(EF) \boxtimes \rho(\mathbf{x} \xrightarrow{j} \mathbf{x})) \cdot \rho(EF\mathbf{x} \xrightarrow{j-1} I\mathbf{x}) \text{ for } j>1.
\end{aligned}
$$

(C) The relations consist of 4 groups:

(1) Commutativity from $A \otimes R_n$:

$$(a_1 \boxtimes r_1) \cdot (a_2 \boxtimes r_2) = (a_1' \boxtimes r_1') \cdot (a_2' \boxtimes r_2'),$$

for $a_1 \cdot a_2 = a_1' \cdot a_2' \in A,\ r_1 \cdot r_2 = r_1' \cdot r_2' \in R_n$ and

$$(a_1, r_1, a_2, r_2) \neq (\rho(I,EF), e(\mathbf{x}), e(EF), \rho(\mathbf{x} \xrightarrow{i} \mathbf{x})) \text{ if } x_i = n-k+i;$$
$$(a_1, r_1, a_2, r_2) \neq (\rho(EF,I), e(\mathbf{x}), e(I), \rho(\mathbf{x} \xrightarrow{j} \mathbf{x})) \text{ if } x_j = j.$$

(2) Relations for $\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x})$:

$$(e(I) \boxtimes e(\mathbf{x})) \cdot \rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}) = \rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}) \cdot (e(EF) \boxtimes e(\mathbf{x})) = \rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x});$$
$$\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}) \cdot (e(EF) \boxtimes \rho(\mathbf{x} \xrightarrow{i'} \mathbf{x})) = (e(I) \boxtimes \rho(\mathbf{x} \xrightarrow{i'} \mathbf{x})) \cdot \rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}) \text{ if } i \neq i'+1;$$
$$\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}) \cdot (e(EF) \boxtimes r(\mathbf{x} \xrightarrow{i',s_1,\mathbf{v}} \mathbf{y})) = (e(I) \boxtimes r(\mathbf{x} \xrightarrow{i',s_1,\mathbf{v}} \mathbf{y})) \cdot \rho(I\mathbf{y} \xrightarrow{i} EF\mathbf{y});$$
$$\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}) \cdot (\rho(EF,I) \boxtimes e(\mathbf{x})) = 0.$$

(3) Relations for $\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})$:

$$(e(EF) \boxtimes e(\mathbf{x})) \cdot \rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}) = \rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}) \cdot (e(I) \boxtimes e(\mathbf{x})) = \rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x});$$

$$\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}) \cdot (e(I) \boxtimes \rho(\mathbf{x} \xrightarrow{j'} \mathbf{x})) = (e(EF) \boxtimes \rho(\mathbf{x} \xrightarrow{j'} \mathbf{x})) \cdot \rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}) \text{ if } j \neq j' - 1;$$

$$\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}) \cdot (e(I) \boxtimes r(\mathbf{x} \xrightarrow{j', s_1, \mathbf{v}} \mathbf{y})) = (e(EF) \boxtimes r(\mathbf{x} \xrightarrow{j', s_1, \mathbf{v}} \mathbf{y})) \cdot \rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x});$$

$$\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}) \cdot (\rho(I, EF) \boxtimes e(\mathbf{x})) = 0.$$

(4) Relations for $\rho(EF\mathbf{x} \xrightarrow{i} I\mathbf{x})$ and $\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})$:

$$\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}) \cdot \rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}) = 0.$$

(D) The grading $\deg = (\deg_h, \deg_t)$ is defined by:

$$\begin{aligned} \deg_h(a \boxtimes r) &= \deg_h(a) + \deg_h(r) + 2k \deg_t(a) \\ \deg_t(a \boxtimes r) &= n \deg_t(a) + \deg_t(r). \end{aligned}$$

for $a \in A, r \in R_{n,k}$ and

$$\begin{aligned} \deg(\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x})) &= (-2(k-i+1), -(k-i+1)), \\ \deg(\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})) &= (2k+1-2j, n-j). \end{aligned}$$

for $\mathbf{x} \in \mathcal{B}_{n,k}$.

Remark 5.18. In Definition (B) on the differential, Equations (6) and (7) come from the differentials in A and R_n, respectively. The right-hand sides of Equations (8) and (9) are zero if we change the symbol $\boxtimes$ to $\otimes$. In other words, both equations are deformations of the relations in $A \otimes R_n$.

It is not hard to prove that the DG algebra $A \boxtimes R_n$ is formal.

Lemma 5.19. *The t-graded DG algebra $A \boxtimes R_n$ is quasi-isomorphic to its cohomology $A \otimes R_n$.*

Consider a collection of projective DG $A \boxtimes R_n$-modules

$$\{P(\Gamma, \mathbf{x}) = (A \boxtimes R_n) \cdot (e(\Gamma) \boxtimes e(\mathbf{x})) \mid \Gamma \in \mathcal{B}, \mathbf{x} \in \mathcal{B}_n\},$$

and a collection of projective DG $A \otimes R_n$-modules

$$\{PH(\Gamma, \mathbf{x}) = (A \otimes R_n) \cdot (e(\Gamma) \otimes e(\mathbf{x})) \mid \Gamma \in \mathcal{B}, \mathbf{x} \in \mathcal{B}_n\}.$$

Definition 5.20. (1) Let $DGP(A \boxtimes R_n)$ be the smallest full subcategory of $DG(A \boxtimes R_n)$ which contains the projective DG $A \boxtimes R_n$-modules $\{P(\Gamma, \mathbf{x}) \mid \Gamma \in \mathcal{B}, \mathbf{x} \in \mathcal{B}_n\}$ and is closed under the cohomological grading shift functor $[1]$, the t-grading shift functor $\{1\}$ and taking mapping cones.

(2) Let $DGP(A \otimes R_n)$ be the smallest full subcategory of $DG(A \otimes R_n)$ which contains the projective DG $A \otimes R_n$-modules $\{PH(\Gamma, \mathbf{x}) \mid \Gamma \in \mathcal{B}, \mathbf{x} \in \mathcal{B}_n\}$ and is closed under the cohomological grading shift functor $[1]$, the t-grading shift functor $\{1\}$ and taking mapping cones.

Since $A \boxtimes R_n$ is formal, the triangulated categories $H^0(DG(A \boxtimes R_n))$ and $H^0(DG(A \otimes R_n))$ are equivalent. There is an equivalence of their subcategories.

Lemma 5.21. *The triangulated categories $H^0(DGP(A\boxtimes R_n))$ and $H^0(DGP(A\otimes R_n))$ are equivalent. Hence, there are isomorphisms of $\mathbb{Z}[t^{\pm 1}]$-modules:*

$$K_0(H^0(DGP(A \boxtimes R_n))) \cong K_0(H^0(DGP(A \otimes R_n))) \cong \mathbf{U}_T \otimes_{\{T=t^n\}} V_n.$$

Proof. It is easy to see that $K_0(H^0(DGP(A \boxtimes R_n)))$ is isomorphic to a quotient of

$$\mathbb{Z}[T^{\pm 1}]\langle \mathcal{B} \rangle \times \mathbb{Z}[t^{\pm 1}]\langle \mathcal{B}_n \rangle$$

by the relation $(\Gamma \cdot T, \mathbf{x}) = (\Gamma, t^n \mathbf{x})$ for $\Gamma \in \mathcal{B}$ and $\mathbf{x} \in \mathcal{B}_n$ from the t-grading in $A \boxtimes R_n$:

$$\deg_t(a \boxtimes r) = n \deg_t(a) + \deg_t(r). \quad \square$$

Definition 5.22. Define a tensor product functor

$$\begin{array}{rccccc} \chi_n : & H^0(DGP(A)) & \times & H^0(DGP(R_n)) & \to & H^0(DGP(A \otimes R_n)) \\ & M & , & M' & \mapsto & M \otimes M', \end{array}$$

where the grading of $M \otimes M'$ is given by:

$$\begin{aligned} \deg_t(m \otimes m') &= n \deg_t(m) + \deg_t(m'), \\ \deg_h(m \otimes m') &= \deg_h(m) + \deg_h(m') + 2k \deg_t(m), \end{aligned}$$

for $m \in M$ and $m' \in M'$ in $DGP(R_{n,k})$.

Remark 5.23. The grading on $M \otimes M'$ makes it into a t-graded DG $A \otimes R_n$-module.

6. THE t-GRADED DG $(H(R_n), A \boxtimes R_n)$-BIMODULE C_n

In order to define a functor $DGP(A \boxtimes R_n) \to DGP(H(R_n))$, we construct the t-graded DG $(H(R_n), A \boxtimes R_n)$-bimodule C_n in 4 steps:

(1) We define the first part of the left $H(R_n)$-module C_n corresponding to the categorical action of I, E, F on the objects of $DGP(R_n)$ in Section 6.1.
(2) We define the first part of the right $A \boxtimes R_n$-module structure on C_n corresponding to the categorical action of I, E, F on the morphisms of $DGP(R_n)$ in Section 6.2.
(3) We finish the construction of the left $H(R_n)$-module C_n corresponding to the action of EF in Section 6.3.
(4) We finish the definition of the right $A \boxtimes R_n$-module structure on C_n corresponding to the action of EF in Section 6.4.

The algebraic construction is quite technical, but the geometric interpretation in terms of decorated rook diagrams is easy to follow.

6.1. **The left DG $H(R_n)$-module C_n, Part I.** As a left DG $H(R_n)$-module,

$$C_n = \bigoplus_{\Gamma \in \mathcal{B}, \mathbf{x} \in \mathcal{B}_n} C_n(\Gamma, \mathbf{x})$$

In this subsection we define $C_n(\Gamma, \mathbf{x})$ for $\Gamma \in \{I, E, F\}$ and $\mathbf{x} \in \mathcal{B}_n$. We fix some $n > 0$ throughout this section and omit the subscript n.

6.1.1. *The case* $\Gamma = I$. Define $C(I, \mathbf{x}) = PH(\mathbf{x}) \in DGP(H(R_n))$ for all $\mathbf{x} \in \mathcal{B}_n$.

6.1.2. *The case* $\Gamma = F$. For $\mathbf{x} \in \mathcal{B}_{n,k}$, define

$$C(F, \mathbf{x}) = \bigoplus_{j=1}^{n-k} PH(\mathbf{x} \sqcup \{\bar{x}_j\})\{n - \bar{x}_j\}[\beta(\mathbf{x}, \bar{x}_j)]$$

with a differential $d(F, \mathbf{x}) = \sum\limits_{j=2}^{n-k} d_j(F, \mathbf{x})$, where

$$d_j(F, \mathbf{x}) : PH(\mathbf{x} \sqcup \{\bar{x}_j\})\{n - \bar{x}_j\}[\beta(\mathbf{x}, \bar{x}_j)] \xrightarrow{\cdot r_F(\mathbf{x};j)} PH(\mathbf{x} \sqcup \{\bar{x}_{j-1}\})\{n - \bar{x}_{j-1}\}[\beta(\mathbf{x}, \bar{x}_{j-1})]$$

is defined below for $2 \leq j \leq n - k$.

Let $(F\mathbf{x})_j$ denote $\mathbf{x} \sqcup \{\bar{x}_j\}$ in $\Gamma_{n,k+1}$ and let

$$q_j = |l \in \{1, \ldots, k\} \mid x_l < \bar{x}_j\}| \,,$$

for $1 \leq j \leq n-k$. The number of $|1\rangle$ states between $\bar{x}_{j-1}$ and $\bar{x}_j$ is $q_j - q_{j-1}$. Recall that $\mathbf{x} \xrightarrow{i} \mathbf{y}$ is the shorthand for the arrow $\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y}$ with $s_1 = s_0(\mathbf{v}) = 0$ in the quiver Q_n. Then there exists a path:

$$(F\mathbf{x})_j \xrightarrow{q_{j-1}+1} \mathbf{z}^1 \xrightarrow{q_{j-1}+2} \cdots \xrightarrow{q_j} \mathbf{z}^{q_j-q_{j-1}} \xrightarrow{q_j+1} (F\mathbf{x})_{j-1}.$$

Define $r_F(\mathbf{x};j) \in H(R_{n,k+1})$ as the product of the corresponding $q_j - q_{j-1} + 1$ generators:

$$r_F(\mathbf{x};j) = r((F\mathbf{x})_j \xrightarrow{q_{j-1}+1} \mathbf{z}^1) \cdots r(\mathbf{z}^{q_j-q_{j-1}} \xrightarrow{q_j+1} (F\mathbf{x})_{j-1}).$$

The differential $d_j(F,\mathbf{x})$ is a map of left $H(R_n)$-modules given by right multiplication by $r_F(\mathbf{x};j)$:

$$d_j(F,\mathbf{x})(m((F\mathbf{x})_j)) = r_F(\mathbf{x};j) \cdot m((F\mathbf{x})_{j-1}).$$

Here $m((F\mathbf{x})_j) \in PH((F\mathbf{x})_j)\{n-\bar{x}_j\}[\beta(\mathbf{x},\bar{x}_j)]$ is the generator of the left projective $H(R_n)$-module for $1 \leq j \leq n-k$. We verify that $d(F,\mathbf{x})$ is a differential in the following lemma. Hence,

$$(C(F,\mathbf{x}), d(F,\mathbf{x})) \in DGP(H(R_n)).$$

Remark 6.1. The definition of the left $H(R_n)$-module $C(F,\mathbf{x})$ comes from a projective resolution of the left $H(R_n)$-module which corresponds to the dividing set $F \cdot \mathbf{x}$ in the contact category $\tilde{\mathcal{C}}_n$. Other left $H(R_n)$-modules $C(\Gamma,\mathbf{x})$ are defined in a similar way.

Lemma 6.2. *$d_j(F,\mathbf{x})$ is a map of degree $(1,0)$ and $d_{j-1} \circ d_j = 0$.*

Proof. (1) The degrees of the generators are as follows:

$$\begin{aligned} \deg(m((F\mathbf{x})_j)) &= (-\beta(\mathbf{x},\bar{x}_j), \bar{x}_j - n), \\ \deg(m((F\mathbf{x})_{j-1})) &= (-\beta(\mathbf{x},\bar{x}_{j-1}), \bar{x}_{j-1} - n), \\ \deg(r_F(\mathbf{x};j)) &= (q_j - q_{j-1} + 1, q_j - q_{j-1} + 1). \end{aligned}$$

Since $\beta(\mathbf{x},\bar{x}_{j-1}) - \beta(\mathbf{x},\bar{x}_j) = q_j - q_{j-1}$ and $(n-\bar{x}_{j-1}) - (n-\bar{x}_j) = q_j - q_{j-1} + 1$, we have

$$\deg(m((F\mathbf{x})_j)) - \deg(m((F\mathbf{x})_{j-1})) = \deg(r_F(\mathbf{x};j)) - (1,0)$$

which implies that $d_j(F,\mathbf{x})$ is a map of degree $(1,0)$.

(2) For a diagrammatic proof of $d_{j-1} \circ d_j = 0$, see Figure 14. The composition $d_{j-1} \circ d_j$ is right multiplication by $r_F(\mathbf{x};j) \cdot r_F(\mathbf{x};j-1)$ and is induced by the following path:

$$\begin{aligned} &(F\mathbf{x})_j \xrightarrow{q_{j-1}+1} \mathbf{z}^1 \to \cdots \to \mathbf{z}^{q_j-q_{j-1}} \xrightarrow{q_j+1} (F\mathbf{x})_{j-1} \\ &\xrightarrow{q_{j-2}+1} \mathbf{w}^1 \to \cdots \to \mathbf{w}^{q_{j-1}-q_{j-2}} \xrightarrow{q_{j-1}+1} (F\mathbf{x})_{j-2}. \end{aligned} \tag{12}$$

By using the commutation relation (R2) to rearrange the arrows, Equation (12) can be written as:

$$(F\mathbf{x})_j \xrightarrow{q_{j-2}+1} \mathbf{u}^1 \to \cdots \to \mathbf{u}^{q_{j-1}-q_{j-2}} \xrightarrow{q_{j-1}+1} \mathbf{v}^0 \xrightarrow{q_{j-1}+1} \mathbf{v}^1 \to \cdots \to \mathbf{v}^{q_j-q_{j-1}} \xrightarrow{q_j+1} (F\mathbf{x})_{j-2}.$$

Hence,

$$r_F(\mathbf{x};j)\cdot r_F(\mathbf{x};j-1) = \cdots r(\mathbf{u}^{q_{j-1}-q_{j-2}} \xrightarrow{q_{j-1}+1} \mathbf{v}^0)\cdot r(\mathbf{v}^0 \xrightarrow{q_{j-1}+1} \mathbf{v}^1)\cdots = 0$$

by Relation (R1) and $d_{j-1}\circ d_j = 0$. □

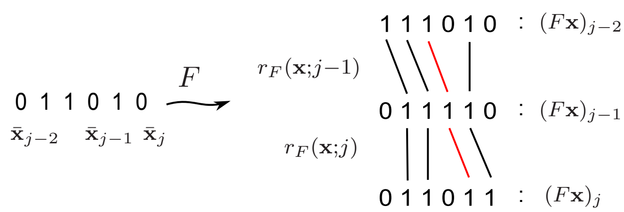


FIGURE 14. A local diagram of $r_F(\mathbf{x};j)\cdot r_F(\mathbf{x};j-1)=0$.

6.1.3. *The case* $\Gamma = E$. For $\mathbf{x}\in\mathcal{B}_{n,k}$, define

$$C(E,\mathbf{x}) = \bigoplus_{i=1}^{k}(PH(\mathbf{x}\backslash\{x_i\})[1-i]\oplus PH(\mathbf{x}\backslash\{x_i\})\{1\}[2-i])$$

with a differential $d(E,\mathbf{x}) = \sum_{i=1}^{k-1} d^i(E,\mathbf{x})$, where

$$d^i(E,\mathbf{x}): PH(\mathbf{x}\backslash\{x_i\})[1-i]\oplus PH(\mathbf{x}\backslash\{x_i\})\{1\}[2-i] \to PH(\mathbf{x}\backslash\{x_{i+1}\})[-i]\oplus PH(\mathbf{x}\backslash\{x_{i+1}\})\{1\}[1-i]$$

is defined below.

Let $(E\mathbf{x})^i$ denote $\mathbf{x}\backslash\{x_i\}$ in $\Gamma_{n,k-1}$ for $1\le i\le k$. Then there exists a path:

$$(E\mathbf{x})^i \xrightarrow{i} \mathbf{z}^1 \xrightarrow{i} \cdots \xrightarrow{i} \mathbf{z}^{x_{i+1}-x_i-1} \xrightarrow{i} (E\mathbf{x})^{i+1}.$$

Define $r_E(\mathbf{x};i)\in H(R_{n,k-1})$ as the product of the generators corresponding to the $x_{i+1}-x_i$ arrows in the path and the $x_{i+1}-x_i-1$ loops attached at the vertices $\mathbf{z}^s$:

$$\begin{aligned} r_E(\mathbf{x};i) =& r((E\mathbf{x})^i \xrightarrow{i} \mathbf{z}^1)\cdot\rho(\mathbf{z}^1 \xrightarrow{i} \mathbf{z}^1)\cdot r(\mathbf{z}^1 \xrightarrow{i} \mathbf{z}^2)\cdots r(\mathbf{z}^{x_{i+1}-x_i-2} \xrightarrow{i} \mathbf{z}^{x_{i+1}-x_i-1}) \\ &\cdot\rho(\mathbf{z}^{x_{i+1}-x_i-1} \xrightarrow{i} \mathbf{z}^{x_{i+1}-x_i-1})\cdot r(\mathbf{z}^{x_{i+1}-x_i-1} \xrightarrow{i} (E\mathbf{x})^{i+1}). \end{aligned}$$

Define loops $\theta(\mathbf{x}; i)$ and $\sigma(\mathbf{x}; i)$ by

$$\theta(\mathbf{x}; i) = \rho((E\mathbf{x})^i \xrightarrow{i} (E\mathbf{x})^i);$$
$$\sigma(\mathbf{x}; i) = \rho((E\mathbf{x})^{i+1} \xrightarrow{i} (E\mathbf{x})^{i+1}).$$

Let $m((E\mathbf{x})^i) \in PH(\mathbf{x}\backslash\{x_i\})[1-i]$ and $m'((E\mathbf{x})^i) \in PH(\mathbf{x}\backslash\{x_i\})\{1\}[2-i]$ be the generators of the left $H(R_n)$ modules for $1 \le i \le k$. Then the differential $d^i(E, \mathbf{x})$ is a map of left $H(R_n)$-modules defined on the generators by:

$$d^i(E,\mathbf{x})(m((E\mathbf{x})^i)) = \theta(\mathbf{x}; i) \cdot r_E(\mathbf{x}; i) \cdot m((E\mathbf{x})^{i+1}) + r_E(\mathbf{x}; i) \cdot m'((E\mathbf{x})^{i+1});$$
$$d^i(E,\mathbf{x})(m'((E\mathbf{x})^i)) = \theta(\mathbf{x}; i) \cdot r_E(\mathbf{x}; i) \cdot \sigma(\mathbf{x}; i) \cdot m((E\mathbf{x})^{i+1}) + r_E(\mathbf{x}; i) \cdot \sigma(\mathbf{x}; i) \cdot m'((E\mathbf{x})^{i+1}).$$

We verify that $d(E, \mathbf{x})$ is a differential in the following lemma. Hence,

$$(C(E,\mathbf{x}), d(E,\mathbf{x})) \in DGP(H(R_n)).$$

Lemma 6.3. *$d(E, \mathbf{x})$ is a map of degree $(1, 0)$ and $d^{i+1} \circ d^i = 0$.*

Proof. (1) It is easy to verify that $d(E, \mathbf{x})$ is a map of degree $(1, 0)$ since

$$\deg(r_E(\mathbf{x}; i)) = (1, 1), \quad \deg(\theta(\mathbf{x}; i)) = (-1, -1), \quad \deg(\sigma(\mathbf{x}; i)) = (-1, -1).$$

(2) We show that $d^{i+1}(d^i(m((E\mathbf{x})^i))) = 0$ and leave the case of $m'((E\mathbf{x})^i))$ to the reader.

$$\begin{aligned}
&d^{i+1}(d^i(m((E\mathbf{x})^i))) \\
=&d^{i+1}\left(\theta(\mathbf{x}; i) \cdot r_E(\mathbf{x}; i) \cdot m((E\mathbf{x})^{i+1}) + r_E(\mathbf{x}; i) \cdot m'((E\mathbf{x})^{i+1})\right) \\
=&\theta(\mathbf{x}; i) \cdot r_E(\mathbf{x}; i) \cdot d^{i+1}(m((E\mathbf{x})^{i+1})) + r_E(\mathbf{x}; i) \cdot d^{i+1}(m'((E\mathbf{x})^{i+1})) \\
=&(\theta(\mathbf{x}; i) \cdot r_E(\mathbf{x}; i) \cdot r_E(\mathbf{x}; i+1) + r_E(\mathbf{x}; i) \cdot r_E(\mathbf{x}; i+1) \cdot \sigma(\mathbf{x}; i+1)) \cdot m'((E\mathbf{x})^{i+2}) \\
&+ \theta(\mathbf{x}; i) \cdot r_E(\mathbf{x}; i) \cdot \theta(\mathbf{x}; i+1) \cdot r_E(\mathbf{x}; i+1) \cdot m((E\mathbf{x})^{i+2}) \\
&+ r_E(\mathbf{x}; i) \cdot \theta(\mathbf{x}; i+1) \cdot r_E(\mathbf{x}; i+1) \cdot \sigma(\mathbf{x}; i+1) \cdot m((E\mathbf{x})^{i+2}).
\end{aligned}$$

We compute the coefficient of $m'((E\mathbf{x})^{i+2})$ in Figure 15, where the diagram representing $r_E(\mathbf{x}; i)$ is defined in Figure 10 in Section 5.1.4. The coefficient is zero by Relations (R2) and (R3). Similarly, we can prove that the coefficient of $m((E\mathbf{x})^{i+2})$ is zero. Hence $d^{i+1}(d^i(m((E\mathbf{x})^i))) = 0$. $\square$

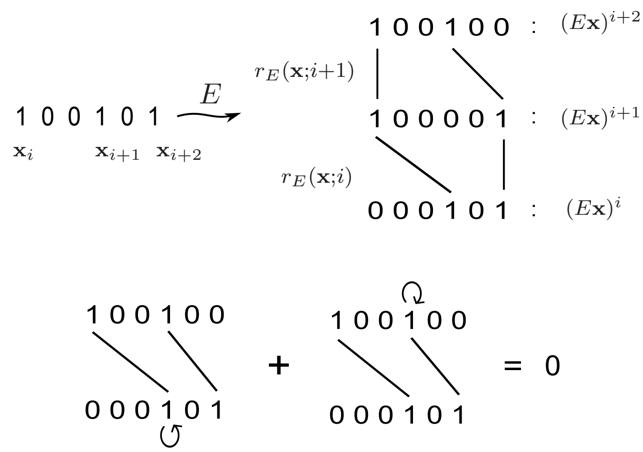


FIGURE 15. The top diagram describes $(E\mathbf{x})^i$ and the bottom diagram represents the coefficient $\theta(\mathbf{x};i)\cdot r_E(\mathbf{x};i)\cdot r_E(\mathbf{x};i+1)+r_E(\mathbf{x};i)\cdot r_E(\mathbf{x};i+1)\cdot\sigma(\mathbf{x};i+1)$.

6.2. **The right $A\boxtimes R_n$-module C_n, Part I.** In this subsection we define the right multiplication by the idempotents $e(\Gamma)\boxtimes e(\mathbf{x})$ and generators $e(\Gamma)\boxtimes\rho(\mathbf{x}\xrightarrow{i}\mathbf{x}), e(\Gamma)\boxtimes r(\mathbf{x}\xrightarrow{i,s_1,\mathbf{v}}\mathbf{y})$ of $A\boxtimes R_n$ for $\Gamma\in\{I,E,F\}$ and $\mathbf{x}\in\mathcal{B}_n$. Let $m\times(a\boxtimes r)$ denote the right multiplication for $m\in C, a\in A, r\in R_n$ and $m\cdot r'$ denote the multiplication in $H(R_n)$ for $m\in PH(\mathbf{x})\subset R_n$ and $r'\in H(R_n)$. Let $j(\mathbf{x},i)$ be the number in $\{0,1,\dots,n-k\}$ such that $\bar{x}_{j(\mathbf{x},i)}<x_i<\bar{x}_{j(\mathbf{x},i)+1}$ for $\mathbf{x}=(x_1,\dots,x_k)\in\mathcal{B}_{n,k}$. Let j_0 denote $j(\mathbf{x},i)$ when $\mathbf{x}$ is understood.

6.2.1. *Idempotents.* Let $a\boxtimes r=e(\Gamma)\boxtimes e(\mathbf{x})$ be an idempotent. Then define

$$m\times(e(\Gamma)\boxtimes e(\mathbf{x}))=\delta_{\Gamma,\Gamma'}\,\delta_{\mathbf{x},\mathbf{x}'}\,m,$$

for $m\in C(\Gamma',\mathbf{x}')$.

6.2.2. *The case $a=e(I)$.* Let $a\boxtimes r=e(I)\boxtimes r$ for $r\in\{\rho(\mathbf{x}\xrightarrow{i}\mathbf{x}),r(\mathbf{x}\xrightarrow{i,s_1,\mathbf{v}}\mathbf{y})\}$. Then define

$$m\times(e(I)\boxtimes r)=\begin{cases} m\cdot\rho(\mathbf{x}\xrightarrow{i}\mathbf{x}) & \text{if } r=\rho(\mathbf{x}\xrightarrow{i}\mathbf{x}),\\ m\cdot r(\mathbf{x}\xrightarrow{i}\mathbf{y}) & \text{if } r=r(\mathbf{x}\xrightarrow{i,s_1,\mathbf{v}}\mathbf{y}), s_1=s_0(\mathbf{v})=0,\\ 0 & \text{otherwise,}\end{cases}$$

for $m\in C(I,\mathbf{x})=PH(\mathbf{x})$. Roughly speaking, we use the quasi-isomorphism $R_n\to H(R_n)$.

6.2.3. *The case* $a = e(F)$.

(1) Let $a \boxtimes r = e(F) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x})$. The right multiplication is a map of left $H(R_n)$-modules: $C(F, \mathbf{x}) \to C(F, \mathbf{x})$, i.e.,

$$\bigoplus_{j=1}^{n-k} PH((F\mathbf{x})_j)\{n - \bar{x}_j\}[\beta(\mathbf{x}, \bar{x}_j)] \xrightarrow{\times(e(F)\boxtimes\rho(\mathbf{x}\xrightarrow{i}\mathbf{x}))} \bigoplus_{j=1}^{n-k} PH((F\mathbf{x})_j)\{n - \bar{x}_j\}[\beta(\mathbf{x}, \bar{x}_j)]$$

defined on the generators by:

$$m((F\mathbf{x})_j) \times (e(F) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x})) = \begin{cases} \rho((F\mathbf{x})_j \xrightarrow{i} (F\mathbf{x})_j) \cdot m((F\mathbf{x})_j) & \text{if } j > j_0; \\ \rho((F\mathbf{x})_j \xrightarrow{i+1} (F\mathbf{x})_j) \cdot m((F\mathbf{x})_j) & \text{if } j \leq j_0. \end{cases}$$

Remark 6.4. The morphism $e(F) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}) \in \mathrm{Hom}_{\tilde{\mathcal{C}}_n}(F \cdot \mathbf{x}, F \cdot \mathbf{x})$ represents a tight contact structure from the dividing curve $F \cdot \mathbf{x}$ to itself. Recall from Remark 6.1 that $(C(F, \mathbf{x}), d(F, \mathbf{x}))$ is the "projective resolution" of $F \cdot \mathbf{x}$. Then the right multiplication $C(F, \mathbf{x}) \to C(F, \mathbf{x})$ is the expression of the morphism between their projective resolutions. Right multiplications by other generators are defined in a similar way.

Lemma 6.5. *The right multiplication by* $e(F)\boxtimes\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})$ *is compatible with the relation in* $A\boxtimes R_n$*:*

$$(m\times(e(F)\boxtimes\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})))\times(e(F)\boxtimes\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})) = m\times((e(F)\boxtimes\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})))\cdot(e(F)\boxtimes\rho(\mathbf{x} \xrightarrow{i} \mathbf{x})) = 0.$$

Proof. It follows from the following diagram where we are ignoring grading shifts on the modules:

$$\begin{tikzcd} \cdots \arrow[r] & PH((F\mathbf{x})_{j_0+1}) \arrow[r, "d"] & PH((F\mathbf{x})_{j_0}) \arrow[r] & \cdots \\ \cdots \arrow[r] & PH((F\mathbf{x})_{j_0+1}) \arrow[r, "d"] \arrow[u, "\cdot\rho((F\mathbf{x})_{j_0+1}\xrightarrow{i}(F\mathbf{x})_{j_0+1})"] & PH((F\mathbf{x})_{j_0}) \arrow[r] \arrow[u, "\cdot\rho((F\mathbf{x})_{j_0}\xrightarrow{i+1}(F\mathbf{x})_{j_0})"'] & \cdots \end{tikzcd}$$

since $\rho((F\mathbf{x})_j \xrightarrow{i} (F\mathbf{x})_j) \cdot \rho((F\mathbf{x})_j \xrightarrow{i} (F\mathbf{x})_j) = 0$ for all i and j. □

Lemma 6.6. *The right multiplication by* $e(F) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x})$ *commutes with the differential.*

Proof. The commutativity for each square which is not in the diagram above follows from Commutativity Relation (R2) since the corresponding decorated rook diagrams are disjoint. The commutativity for the square follows from Relation (R3) since the corresponding decorated rook diagrams arise as the resolution of a crossing as shown in Figure 16. □

(2) Let $a \boxtimes r = e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ with $s_1 = s_0(\mathbf{v}) = 0$ for $\mathbf{x}, \mathbf{y} \in Q_{n,k}$. Note that

$$\bar{y}_{j_0} = \bar{x}_{j_0} + 1; \quad \bar{y}_j = \bar{x}_j \text{ for } j \neq j_0.$$

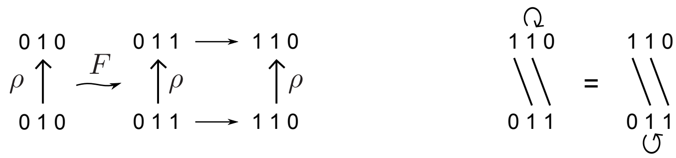


FIGURE 16. The left-hand diagram represents the right multiplication by $e(F) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x})$; the right-hand diagram shows that it commutes with the differential.

Then we have $(F\mathbf{x})_{j_0} = (F\mathbf{y})_{j_0} \in \Gamma_{n,k+1}$ and there exist arrows in $\Gamma_{n,k+1}$:

$$(F\mathbf{x})_j \xrightarrow{i} (F\mathbf{y})_j \text{ for } j > j_0,$$

$$(F\mathbf{x})_j \xrightarrow{i+1} (F\mathbf{y})_j \text{ for } j < j_0.$$

The right multiplication is a map of left $H(R_n)$-modules defined on the generators by:

$$m((F\mathbf{x})_j) \times (e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})) = \begin{cases} r((F\mathbf{x})_j \xrightarrow{i} (F\mathbf{y})_j) \cdot m((F\mathbf{y})_j) & \text{if } j > j_0; \\ m((F\mathbf{y})_j) & \text{if } j = j_0; \\ r((F\mathbf{x})_j \xrightarrow{i+1} (F\mathbf{y})_j) \cdot m((F\mathbf{y})_j) & \text{if } j < j_0. \end{cases}$$

See the left-hand diagram in Figure 17 for an example.

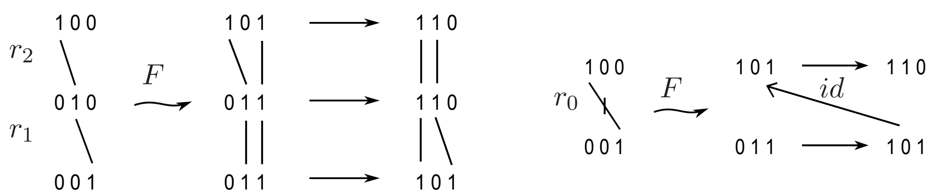


FIGURE 17. The right multiplications by $e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,0,(0)} \mathbf{y})$ and $e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,0,(1)} \mathbf{y})$ on the left and right, respectively.

Lemma 6.7. *The right multiplication by $e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ such that $s_1 = s_0(\mathbf{v}) = 0$ commutes with the differential.*

Proof. We have the following diagram

$$\begin{tikzcd} PH((F\mathbf{y})_{j_0+1}) \arrow[r, "d"] & PH((F\mathbf{y})_{j_0}) \arrow[r, "d"] & PH((F\mathbf{y})_{j_0-1}) \\ PH((F\mathbf{x})_{j_0+1}) \arrow[r, "d"] \arrow[u, "\cdot r((F\mathbf{x})_{j_0+1} \xrightarrow{i} (F\mathbf{y})_{j_0+1})"] & PH((F\mathbf{x})_{j_0}) \arrow[r, "d"] \arrow[u, "id"'] & PH((F\mathbf{x})_{j_0-1}) \arrow[u, "\cdot r((F\mathbf{x})_{j_0-1} \xrightarrow{i+1} (F\mathbf{y})_{j_0-1})"'] \end{tikzcd}$$

The commutativity follows from the commutativity relation (R2). □

(3) Let $a \boxtimes r = e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ with $s_1 = 0, s_0(\mathbf{v}) > 0$ for $\mathbf{x}, \mathbf{y} \in Q_{n,k}$. Let s_0 denote $s_0(\mathbf{v})$ for simplicity. Note that $(F\mathbf{x})_{j_0 - s_0} = (F\mathbf{y})_{j_0} \in \Gamma_{n,k+1}$. Then the right multiplication is a map of left $H(R_n)$-modules defined on the generators by:

$$m((F\mathbf{x})_j) \times (e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})) = \begin{cases} m((F\mathbf{y})_{j+s_0}) & \text{if } j = j_0 - s_0; \\ 0 & \text{otherwise}. \end{cases}$$

See the right-hand diagram in Figure 17 for an example.

We verify that the definition is compatible with the DG structure on $A \boxtimes R_n$.

Lemma 6.8. *$d(m \times r) = d(m) \times r + m \times d(r)$ holds for $r = e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,0,\mathbf{v}} \mathbf{y})$.*

Proof. (1) The case for $s_0(\mathbf{v}) = 0$ is proved in Lemma 6.7.

(2) For $s_0(\mathbf{v}) > 0$, a diagrammatic proof for $r_0 = e(F) \boxtimes r(\mathbf{x} \xrightarrow{1,0,(1)} \mathbf{y})$ is given in Figure 17, where $\mathbf{x} = (3) = |001\rangle, \mathbf{y} = (1) = |100\rangle \in \mathcal{B}_{3,1}$. Recall that

$$d(r_0) = e(F) \boxtimes r(\mathbf{x} \xrightarrow{1,0,(0)} \mathbf{z}) \cdot e(F) \boxtimes r(\mathbf{z} \xrightarrow{1,0,(0)} \mathbf{y}) = r_1 \cdot r_2,$$

where $\mathbf{z} = (2) = |010\rangle$. In Figure 17, the right multiplications by r_1, r_2 and r_0 are given in the left-hand and right-hand diagrams, respectively.

We verify the equation for $m = m(|011\rangle) \in C(F, |001\rangle)$ by chasing the diagrams and leave other cases to the reader. The right-hand side of the equation is zero since

$$\begin{aligned} d(m(|011\rangle)) \times r_0 =& r(|011\rangle \xrightarrow{1} |101\rangle) \cdot (m(|101\rangle) \times r_0) \\ =& r(|011\rangle \xrightarrow{1} |101\rangle) \cdot m(|101\rangle) \in C(F, |100\rangle), \\ m(|011\rangle) \times d(r_0) =& (m(|011\rangle) \times r_1) \times r_2 = m(|011\rangle) \times r_2 \\ =& r(|011\rangle \xrightarrow{1} |101\rangle) \cdot m(|101\rangle) \in C(F, |100\rangle). \end{aligned}$$

The left-hand side is obviously zero since $m(|011\rangle) \times r_0 = 0$.

The proof for the case $s_0(\mathbf{v}) > 0$ in general is similar. □

(4) Let $a \boxtimes r = e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ with $s_1 > 0$ for $\mathbf{x}, \mathbf{y} \in Q_{n,k}$. The right multiplication is defined as the zero map.

6.2.4. *The case $a = e(E)$.*

(1) Let $a \boxtimes r = e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})$. The right multiplication is a map of left $H(R_n)$-modules:

$C(E,\mathbf{x}) \to C(E,\mathbf{x})$ defined on the generators by:

$$m((E\mathbf{x})^i) \times (e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})) = \begin{cases} \rho((E\mathbf{x})^i \xrightarrow{i_0-1} (E\mathbf{x})^i) \cdot m((E\mathbf{x})^i) & \text{if } i < i_0, \\ m'((E\mathbf{x})^i) & \text{if } i = i_0, \\ \rho((E\mathbf{x})^i \xrightarrow{i_0} (E\mathbf{x})^i) \cdot m((E\mathbf{x})^i) & \text{if } i > i_0; \end{cases}$$

$$m'((E\mathbf{x})^i) \times (e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})) = \begin{cases} \rho((E\mathbf{x})^i \xrightarrow{i_0-1} (E\mathbf{x})^i) \cdot m'((E\mathbf{x})^i) & \text{if } i < i_0, \\ 0 & \text{if } i = i_0, \\ \rho((E\mathbf{x})^i \xrightarrow{i_0} (E\mathbf{x})^i) \cdot m'((E\mathbf{x})^i) & \text{if } i > i_0. \end{cases}$$

A diagrammatic example is given in Figure 18, where $|011\rangle$ denotes the first summand $PH((E|111\rangle)^1)$ and $|011\rangle'$ denotes the second summand $PH((E|111\rangle)^1)\{1\}[1]$ in $C(E,|111\rangle)$.

$$\begin{tikzcd}
111 & & 011 & \oplus & 011' \arrow[r] & 101 & \oplus & 101' \arrow[r] & 110 & \oplus & 110' \\
111 \arrow[u, "\rho"] \arrow[r, squiggly, "E"] & {} & 011 \arrow[u, "\rho"] & \oplus & 011' \arrow[u, "\rho"] \arrow[r] & 101 \arrow[urr, "id"] & \oplus & 101' \arrow[r] & 110 \arrow[u, "\rho"'] & \oplus & 110' \arrow[u, "\rho"']
\end{tikzcd}$$

FIGURE 18. The right multiplication by $e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})$.

Lemma 6.9. *The right multiplication by $e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})$ is compatible with the relation in $A \boxtimes R_n$:*

$$(m\times(e(E)\boxtimes\rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})))\times(e(E)\boxtimes\rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})) = m\times((e(E)\boxtimes\rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x}))\cdot(e(E)\boxtimes\rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})) = 0.$$

Proof. It easily follows from Figure 18. □

Lemma 6.10. *The right multiplication by $e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})$ commutes with the differential.*

Proof. It suffices to prove that

$$d(m \times (e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x}))) = d(m) \times (e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})),$$

where $m = m((E\mathbf{x})^i), m'((E\mathbf{x})^i)$ for $1 \leq i \leq k$. The equation is obviously true for $i \neq i_0 - 1, i_0$ from the commutativity relation (R2).

We verify the equation for $m = m((E\mathbf{x})^{i_0})$:

$$\begin{aligned}
&d(m((E\mathbf{x})^{i_0}) \times (e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})))\\
=&d(m'((E\mathbf{x})^{i_0}))\\
=&\theta(\mathbf{x}; i_0) \cdot r_E(\mathbf{x}; i_0) \cdot \sigma(\mathbf{x}; i_0) \cdot m((E\mathbf{x})^{i_0+1}) + r_E(\mathbf{x}; i_0) \cdot \sigma(\mathbf{x}; i_0) \cdot m'((E\mathbf{x})^{i_0+1})\\
=&d(m((E\mathbf{x})^{i_0})) \cdot \rho((E\mathbf{x})^{i_0+1} \xrightarrow{i_0} (E\mathbf{x})^{i_0+1})\\
=&d(m((E\mathbf{x})^{i_0})) \times (e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{i_0} \mathbf{x})).
\end{aligned}$$

The proof for other cases is similar and we leave it to the reader. □

(2) Let $a \boxtimes r = e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0, s_1, \mathbf{v}} \mathbf{y})$ with $s_1 = s_0(\mathbf{v}) = 0$ for $\mathbf{x}, \mathbf{y} \in Q_{n,k}$. Note that

$$y_{i_0} = x_{i_0} - 1; \quad y_i = x_i \text{ for } i \neq i_0.$$

We have $(E\mathbf{x})^{i_0} = (E\mathbf{y})^{i_0} \in \Gamma_{n,k-1}$ and there exist arrows:

$$(E\mathbf{x})^i \xrightarrow{i_0-1} (E\mathbf{y})^i \text{ for } i < i_0,$$
$$(E\mathbf{x})^i \xrightarrow{i_0} (E\mathbf{y})^i \text{ for } i > i_0.$$

Then the right multiplication is a map of left $H(R_n)$-modules defined on the generators by:

$$m((E\mathbf{x})^i) \times (e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0, s_1, \mathbf{v}} \mathbf{y})) = \begin{cases} r((E\mathbf{x})^i \xrightarrow{i_0-1} (E\mathbf{y})^i) \cdot m((E\mathbf{y})^i) & \text{if } i < i_0; \\ 0 & \text{if } i = i_0; \\ r((E\mathbf{x})^i \xrightarrow{i_0} (E\mathbf{y})^i) \cdot m((E\mathbf{y})^i) & \text{if } i > i_0. \end{cases}$$

$$m'((E\mathbf{x})^i) \times (e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0, s_1, \mathbf{v}} \mathbf{y})) = \begin{cases} r((E\mathbf{x})^i \xrightarrow{i_0-1} (E\mathbf{y})^i) \cdot m'((E\mathbf{y})^i) & \text{if } i < i_0; \\ m((E\mathbf{y})^i) & \text{if } i = i_0; \\ r((E\mathbf{x})^i \xrightarrow{i_0} (E\mathbf{y})^i) \cdot m'((E\mathbf{y})^i) & \text{if } i > i_0. \end{cases}$$

A diagrammatic example is given in Figure 19.

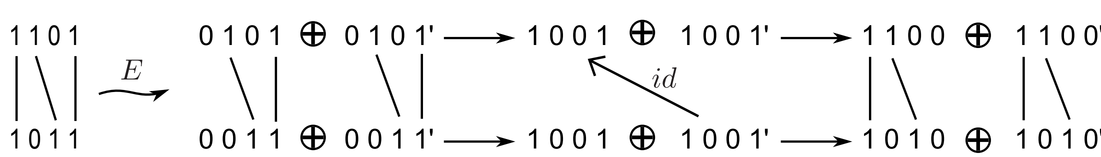


FIGURE 19. The right multiplication by $e(E) \boxtimes r(\mathbf{x} \xrightarrow{i,0,(0)} \mathbf{y})$.

Lemma 6.11. *The right multiplication by* $e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0,s_1,\mathbf{v}} \mathbf{y})$ *such that* $s_1 = s_0(\mathbf{v}) = 0$ *commutes with the differential.*

Proof. It suffices to prove that

$$d(m \times (e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0,s_1,\mathbf{v}} \mathbf{y}))) = d(m) \times (e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0,s_1,\mathbf{v}} \mathbf{y})),$$

where $m = m((E\mathbf{x})^i), m'((E\mathbf{x})^i)$ for $1 \leq i \leq k$. The equation is obviously true for $i \neq i_0 - 1, i_0$ from the commutativity Relation (R2).

We verify the equation for $m = m'((E\mathbf{x})^{i_0})$:

$$\begin{aligned} & d\left(m'((E\mathbf{x})^{i_0}) \times (e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0,s_1,\mathbf{v}} \mathbf{y}))\right) \\ =& d(m((E\mathbf{y})^{i_0})) \\ =& \theta(\mathbf{y}; i_0) \cdot r_E(\mathbf{y}; i_0) \cdot m((E\mathbf{y})^{i_0+1}) + r_E(\mathbf{y}; i_0) \cdot m'((E\mathbf{y})^{i_0+1}) \\ =& d(m'((E\mathbf{x})^{i_0})) \cdot r((E\mathbf{x})^{i_0+1} \xrightarrow{i_0} (E\mathbf{y})^{i_0+1}) \\ =& d(m'((E\mathbf{x})^{i_0})) \times (e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0,s_1,\mathbf{v}} \mathbf{y})). \end{aligned}$$

The proof for other cases is similar and we leave it to the reader. □

(3) Let $a \boxtimes r = e(E) \boxtimes r(\mathbf{x} \xrightarrow{i_0,s_1,\mathbf{v}} \mathbf{y})$ with $s_1 > 0, s_0(\mathbf{v}) = 0$ for $\mathbf{x}, \mathbf{y} \in Q_{n,k}$. Note that $(E\mathbf{x})^{i_0+s_1} = (E\mathbf{y})^{i_0} \in \Gamma_{n,k-1}$. See Figure 20. The right multiplication is a map of left $H(R_n)$-modules defined on the generators by:

$$m \times (e(E) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})) = \begin{cases} m((E\mathbf{y})^{i-s_1}) & \text{if } m = m'((E\mathbf{x})^{i_0+s_1}); \\ 0 & \text{otherwise}. \end{cases}$$

See the right-hand diagram in Figure 20 for an example.

We verify that the definition is compatible with the DG structure on $A \boxtimes R_n$.

Lemma 6.12. $d(m \times r) = d(m) \times r + m \times d(r)$ *holds for* $r = e(E) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ *with* $s_0(\mathbf{v}) = 0$.

Proof. (1) The case for $s_1 = 0$ is proved in Lemma 6.11.

(2) For $s_1 > 0$, a diagrammatic proof for $r_0 = e(E) \boxtimes r(\mathbf{x} \xrightarrow{1,1,(0,0)} \mathbf{y})$ is given in Figure 20, where $\mathbf{x} = (2,3) = |011\rangle, \mathbf{y} = (1,2) = |110\rangle \in \mathcal{B}_{3,2}$. Recall that $d(r_0) = r_1 \cdot r_2 \cdot r_3 + r_2 \cdot r_3 \cdot r_4$, where

$$r_1 = e(E) \boxtimes \rho(\mathbf{x} \xrightarrow{1} \mathbf{x}), \quad r_2 = e(E) \boxtimes r(\mathbf{x} \xrightarrow{1,0,(0)} \mathbf{z}),$$

$$r_4 = e(E) \boxtimes \rho(\mathbf{y} \xrightarrow{2} \mathbf{y}), \quad r_3 = e(E) \boxtimes r(\mathbf{z} \xrightarrow{2,0,(0)} \mathbf{y}),$$

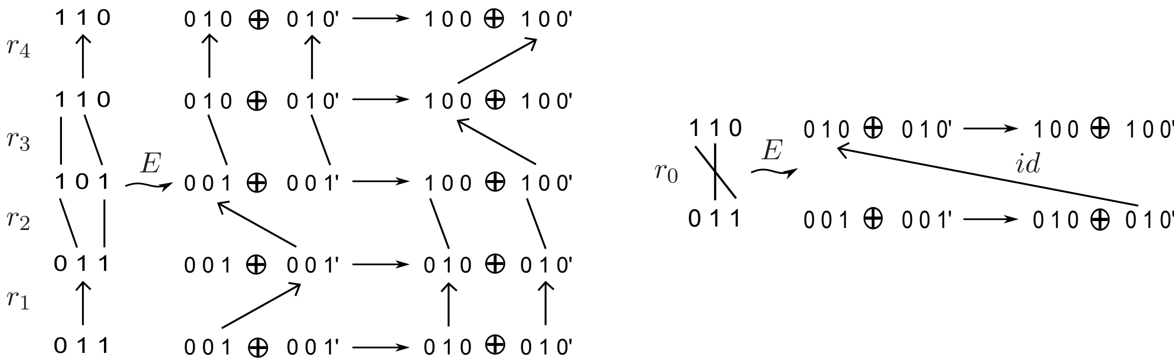


FIGURE 20. The right multiplication by $e(E) \boxtimes r(\mathbf{x} \xrightarrow{i,1,(0,0)} \mathbf{y})$ on the right is compatible with the right multiplication by its differential on the left.

and $\mathbf{z} = (1,3) = |101\rangle$. In Figure 20, the right multiplications by r_1, r_2, r_3, r_4 and r_0 are given in the left-hand and right-hand diagrams, respectively.

We verify the equation for $m = m(|001\rangle) \in C(E, |011\rangle)$ by chasing the diagrams and leave other cases to the reader. The right-hand side of the equation is zero since

$$\begin{aligned}
&d(m(|001\rangle)) \times r_0 \\
=&\Big(\rho(|001\rangle \xrightarrow{1} |001\rangle) \cdot r(|001\rangle \xrightarrow{1} |010\rangle) \cdot m(|010\rangle) + r(|001\rangle \xrightarrow{1} |010\rangle) \cdot m'(|010\rangle)\Big) \times r_0 \\
=&r(|001\rangle \xrightarrow{1} |010\rangle) \cdot (m'(|010\rangle) \times r_0) \\
=&r(|001\rangle \xrightarrow{1} |010\rangle) \cdot m(|010\rangle) \in C(E, |110\rangle),
\end{aligned}$$

is the same as

$$\begin{aligned}
m(|001\rangle) \times d(r_0) =&m(|001\rangle) \times (r_1 \cdot r_2 \cdot r_3 + r_2 \cdot r_3 \cdot r_4) = ((m(|001\rangle) \times r_1) \times r_2) \times r_3 \\
=&r(|001\rangle \xrightarrow{1} |010\rangle) \cdot m(|010\rangle) \in C(E, |110\rangle).
\end{aligned}$$

The left-hand side is obviously zero since $m(|001\rangle) \times r_0 = 0$.

The proof for the case $s_1 > 0$ in general is similar. □

(4) Let $a \boxtimes r = e(E) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ with $s_0(\mathbf{v}) > 0$ for $\mathbf{x}, \mathbf{y} \in Q_{n,k}$. The right multiplication is defined as the zero map.

6.3. **The left DG $H(R_n)$-module C_n, Part II.** We finish the definition of the left $H(R_n)$-module structure on $C(\Gamma, \mathbf{x})$ for $\Gamma = EF$ and $\mathbf{x} \in \mathcal{B}_{n,k}$. The module $C(EF, \mathbf{x})$ is constructed through the

action of E on the module $C(F,\mathbf{x})$. Let $(EF\mathbf{x})^i_j$ denote $(E(F\mathbf{x})_j)^i \in \mathcal{B}_{n,k}$, i.e.,

$$(EF\mathbf{x})^i_j = \begin{cases} \mathbf{x} \sqcup \{\bar{x}_j\} \backslash \{x_i\} & \text{if } i < q_j + 1; \\ \mathbf{x} & \text{if } i = q_j + 1; \\ \mathbf{x} \sqcup \{\bar{x}_j\} \backslash \{x_{i-1}\} & \text{if } i > q_j + 1. \end{cases}$$

Define

$$\begin{aligned} C(EF,\mathbf{x}) &= \bigoplus_{j=1}^{n-k} C(E,(F\mathbf{x})_j)\{n-\bar{x}_j\}[\beta(\mathbf{x},\bar{x}_j)] \\ &= \bigoplus_{j=1}^{n-k}\bigoplus_{i=1}^{k+1} (PH((EF\mathbf{x})^i_j)\{n-\bar{x}_j\}[\beta(\mathbf{x},\bar{x}_j)+1-i] \\ &\qquad \oplus PH((EF\mathbf{x})^i_j)\{n-\bar{x}_j+1\}[\beta(\mathbf{x},\bar{x}_j)+2-i]). \end{aligned}$$

Recall that $r_F(\mathbf{x};j) \in H(R_{n,k+1})$ is given by the path from $(F\mathbf{x})_j$ to $(F\mathbf{x})_{j-1}$. It can also be viewed as an element in $R_{n,k+1}$ which is still denoted by $r_F(\mathbf{x};j)$. Then we have the right multiplication by $e(E) \boxtimes r_F(\mathbf{x};j)$:

$$\times (e(E) \boxtimes r_F(\mathbf{x};j)) : C(E,(F\mathbf{x})_j) \to C(E,(F\mathbf{x})_{j-1}).$$

We view $C(EF,\mathbf{x})$ as a double complex with (i,j)-th entry $C^i_j(EF,\mathbf{x})$ equal to:

$$PH((EF\mathbf{x})^i_j)\{n-\bar{x}_j\}[\beta(\mathbf{x},\bar{x}_j)+1-i] \oplus PH((EF\mathbf{x})^i_j)\{n-\bar{x}_j+1\}[\beta(\mathbf{x},\bar{x}_j)+2-i].$$

Let $m((EF\mathbf{x})^i_j)$ and $m'((EF\mathbf{x})^i_j)$ be the generators of the first and the second summand of $C^i_j(EF,\mathbf{x})$. The differential $d(EF,\mathbf{x})$ is given by

$$d(EF,\mathbf{x}) = \sum_{j=1}^{n-k}\sum_{i=1}^{k+1} d^i_j(EF,\mathbf{x}) = \sum_{j=1}^{n-k}\sum_{i=1}^{k+1} \left(d^i_j|_{ver}(EF,\mathbf{x}) + d^i_j|_{hor}(EF,\mathbf{x})\right),$$

where

$$d^i_j|_{ver}(EF,\mathbf{x}) = d^i(E,(F\mathbf{x})_j) : C^i_j(EF,\mathbf{x}) \to C^{i+1}_j(EF,\mathbf{x});$$

$$d^i_j|_{hor}(EF,\mathbf{x}) = (\times e(E) \boxtimes r_F(\mathbf{x};j)) : C^i_j(EF,\mathbf{x}) \to C^i_{j-1}(EF,\mathbf{x}).$$

We have the following double complex $(C(EF,\mathbf{x}), d(EF,\mathbf{x}))$:

$$
\begin{tikzcd}
C^{k+1}_{n-k}(EF,\mathbf{x}) \arrow[r] & C^{k+1}_{n-k-1}(EF,\mathbf{x}) \arrow[r] & \cdots \arrow[r] & C^{k+1}_{1}(EF,\mathbf{x}) \\
\vdots \arrow[u] \arrow[r] & \vdots \arrow[u] \arrow[r] & \vdots \arrow[u] \arrow[r] & \vdots \arrow[u] \\
C^{1}_{n-k}(EF,\mathbf{x}) \arrow[r] \arrow[u, "d^1(E,(F\mathbf{x})_{n-k})"] & C^{1}_{n-k-1}(EF,\mathbf{x}) \arrow[r] \arrow[u, "d^1(E,(F\mathbf{x})_{n-k-1})"] & \cdots \arrow[r] \arrow[u] & C^{1}_{1}(EF,\mathbf{x}) \arrow[u, "d^1(E,(F\mathbf{x})_{1})"]
\end{tikzcd}
\tag{13}
$$

Lemma 6.13. *The differential $d(EF,\mathbf{x})$ is well-defined.*

Proof. Since $r_F(\mathbf{x};j)$ is a product of the generators which satisfy $s_1 = s_0(\mathbf{v}) = 0$, the horizontal differential $d^i_j|_{hor}(EF,\mathbf{x})$, i.e., the right multiplication by $e(E) \boxtimes r_F(\mathbf{x};j)$, commutes with the vertical differential $d^i_j|_{ver}(EF,\mathbf{x})$ by Lemma 6.11. Hence $d(EF,\mathbf{x}) \circ d(EF,\mathbf{x}) = 0$. $\square$

This concludes the construction of the left $H(R_n)$-module C_n.

The following lemma is immediate:

Lemma 6.14. *If $\Gamma \in \mathcal{B}$ and $\mathbf{x} \in \mathcal{B}_n$, then $[C(\Gamma,\mathbf{x})] = \Gamma(\mathbf{x})$ when viewed as elements in*

$$K_0(H^0(DGP(H(R_n)))) \cong V_n.$$

6.4. **The right $A \boxtimes R_n$-module C_n, Part II.** We finish the definition of the right multiplication by generators in

$$
\begin{gathered}
\{e(EF) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}),\ e(EF) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})\}, \\
\{\rho(I,EF) \boxtimes e(\mathbf{x}),\ \rho(EF,I) \boxtimes e(\mathbf{x}),\},\\
\{\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}), \quad \rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})\},
\end{gathered}
$$

for $\mathbf{x} \in \mathcal{B}_{n,k}$. Because the right multiplication by

$$\rho(I,EF) \boxtimes e(\mathbf{x})) \cdot (e(EF) \boxtimes \rho(\mathbf{x} \xrightarrow{k} \mathbf{x})) + (e(I) \boxtimes \rho(\mathbf{x} \xrightarrow{k} \mathbf{x})) \cdot (\rho(I,EF) \boxtimes e(\mathbf{x})) \tag{14}$$

is possibly nonzero, we represent Equation (14) as the differential of $\rho(I\mathbf{x} \xrightarrow{k} EF\mathbf{x})$ for $\mathbf{x} \in \mathcal{B}_{n,k}$ with $x_k = n$ in Definition 5.17.

6.4.1. *The right multiplication by* $e(EF)\boxtimes r$.

(1) Let $a\boxtimes r = e(EF)\boxtimes \rho(\mathbf{x}\xrightarrow{i}\mathbf{x})$. The right multiplication is a map $C(EF,\mathbf{x})\to C(EF,\mathbf{x})$ of left $H(R_n)$-modules. Recall that

$$C(EF,\mathbf{x}) = \bigoplus_{j=1}^{n-k} C(E,(F\mathbf{x})_j)\{n-\bar{x}_j\}[\beta(\mathbf{x},\bar{x}_j)],$$

and the right multiplication

$$\times(e(F)\boxtimes\rho(\mathbf{x}\xrightarrow{i}\mathbf{x})) : PH((F\mathbf{x})_j)\to PH((F\mathbf{x})_j)$$

is given in Section 6.2.3 (1). Then the right multiplication by $e(EF)\boxtimes\rho(\mathbf{x}\xrightarrow{i}\mathbf{x})$ is defined by:

$$m\times(e(EF)\boxtimes\rho(\mathbf{x}\xrightarrow{i}\mathbf{x})) = \begin{cases} m\times(e(E)\boxtimes\rho((F\mathbf{x})_j\xrightarrow{i}(F\mathbf{x})_j) & \text{if } j>j_0,\\ m\times(e(E)\boxtimes\rho((F\mathbf{x})_j\xrightarrow{i+1}(F\mathbf{x})_j) & \text{if } j\le j_0,\end{cases}$$

where $m\in C(E,(F\mathbf{x})_j)\{n-\bar{x}_j\}[\beta(\mathbf{x},\bar{x}_j)]\subset C(EF,\mathbf{x})$.

(2) Let $a\boxtimes r = e(EF)\boxtimes r(\mathbf{x}\xrightarrow{i,s_1,\mathbf{v}}\mathbf{y})$. The right multiplication

$$\times(e(F)\boxtimes r(\mathbf{x}\xrightarrow{i,s_1,\mathbf{v}}\mathbf{y})) : PH((F\mathbf{x})_j)\to PH((F\mathbf{y})_{j+s_0})$$

is given in Section 6.2.3 (2), (3) and (4).

Then the right multiplication by $e(EF)\boxtimes r(\mathbf{x}\xrightarrow{i,0,(0)}\mathbf{y})$ is defined by:

$$m\times(e(EF)\boxtimes r(\mathbf{x}\xrightarrow{i,0,(0)}\mathbf{y})) = \begin{cases} m\times(e(E)\boxtimes r((F\mathbf{x})_j\xrightarrow{i,0,(0)}(F\mathbf{x})_j)) & \text{if } j>j_0,\\ m\times(e(E)\boxtimes e((F\mathbf{x})_j)) & \text{if } j=j_0,\\ m\times(e(E)\boxtimes r((F\mathbf{x})_j\xrightarrow{i+1,0,(0)}(F\mathbf{x})_j)) & \text{if } j<j_0,\end{cases}$$

where $m\in C(E,(F\mathbf{x})_j)\{n-\bar{x}_j\}[\beta(\mathbf{x},\bar{x}_j)]\subset C(EF,\mathbf{x})$.

The right multiplication by $e(EF)\boxtimes r(\mathbf{x}\xrightarrow{i,0,(s_0)}\mathbf{y})$ is defined by:

$$m\times(e(EF)\boxtimes r(\mathbf{x}\xrightarrow{i,0,(s_0)}\mathbf{y})) = \begin{cases} m\times(e(E)\boxtimes e((F\mathbf{y})_{j+s_0})) & \text{if } j=j_0-s_0;\\ 0 & \text{otherwise}.\end{cases}$$

where $m\in C(E,(F\mathbf{x})_j)\{n-\bar{x}_j\}[\beta(\mathbf{x},\bar{x}_j)]\subset C(EF,\mathbf{x})$.

The right multiplication by $e(EF)\boxtimes r(\mathbf{x}\xrightarrow{i,s_1,\mathbf{v}}\mathbf{y})$ for $s_1>0$ is defined as the zero map.

6.4.2. *The right multiplication by $\rho(I, EF) \boxtimes e(\mathbf{x})$.* We discuss $(EF\mathbf{x})^{k+1}_{n-k} \in \mathcal{B}_{n,k}$ shown in the top left corner $C^{k+1}_{n-k}(EF, \mathbf{x})$ of the double complex (13) depending on $\bar{x}_{n-k} = n$ or $\bar{x}_{n-k} < n$.

(Case 1) Suppose $\bar{x}_{n-k} = n$, i.e., the last state in the tensor product presentation is $|0\rangle$. Then we have $(EF\mathbf{x})^{k+1}_{n-k} = \mathbf{x}$, $\beta(\mathbf{x}, \bar{x}_{n-k}) = k$ and $C^{k+1}_{n-k}(EF, \mathbf{x}) = PH(\mathbf{x}) \oplus PH(\mathbf{x})\{1\}[1]$. The right multiplication

$$\times(\rho(I, EF) \boxtimes e(\mathbf{x})) : C(I, \mathbf{x}) \to C(EF, \mathbf{x})$$

is defined by the identity map from $PH(\mathbf{x}) = C(I, \mathbf{x})$ to $PH(\mathbf{x}) \subset C^{k+1}_{n-k}(EF, \mathbf{x})$. A diagrammatic example is given in Figure 21.

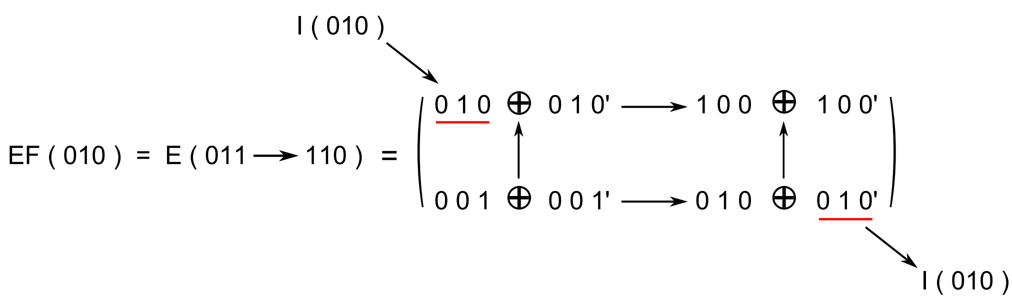


FIGURE 21. The identity map from $I(010)$ to the top left corner is the right multiplication by $\rho(I, EF) \boxtimes e(\mathbf{x})$ when $\bar{x}_{n-k} = n$. The identity map from the bottom right corner to $I(010)$ is the right multiplication by $\rho(EF, I) \boxtimes e(\mathbf{x})$ when $\bar{x}_1 = 1$.

Lemma 6.15. *The right multiplication by $\rho(I, EF) \boxtimes e(\mathbf{x})$ in Case (1) commutes with the differential.*

Proof. It suffices to prove that

$$\begin{aligned} d(m(I\mathbf{x}) \times (\rho(I, EF) \boxtimes e(\mathbf{x}))) &= d^{k+1}_{n-k}|_{hor}(EF, \mathbf{x})(m((EF\mathbf{x})^{k+1}_{n-k})) \\ &= m((EF\mathbf{x})^{k+1}_{n-k}) \times (e(E) \boxtimes r_F(\mathbf{x}; n-k)) = 0 \end{aligned}$$

since the differential on $C(I, \mathbf{x})$ is trivial. Recall that

$$r_F(\mathbf{x}; n-k) = r_0 \cdot r(\mathbf{z} \xrightarrow{q_{n-k}+1} (F\mathbf{x})_{n-k-1})$$

for some $r_0 \in R_{n,k+1}$ and $\mathbf{z} \in \mathcal{B}_{n,k+1}$, where $q_{n-k} = |l \in \{1,...,k\} \mid x_l < \bar{x}_{n-k} = n\}| = k$. Moreover, $(E\mathbf{z})^{k+1} = (E(F\mathbf{x})_{n-k-1})^{k+1} \in \mathcal{B}_{n,k}$. Hence, there exists $r_1 \in R_{n,k}$ such that

$$
\begin{aligned}
& m((EF\mathbf{x})^{k+1}_{n-k}) \times (e(E) \boxtimes r_F(\mathbf{x}; n-k)) \\
=& (m((EF\mathbf{x})^{k+1}_{n-k}) \times (e(E) \boxtimes r_0)) \times (e(E) \boxtimes r(\mathbf{z} \xrightarrow{k+1} (F\mathbf{x})_{n-k-1})) \\
=& (r_1 \cdot m((E\mathbf{z})^{k+1})) \times (e(E) \boxtimes r(\mathbf{z} \xrightarrow{k+1} (F\mathbf{x})_{n-k-1})) \\
=& r_1 \cdot (m((E\mathbf{z})^{k+1}) \times (e(E) \boxtimes r(\mathbf{z} \xrightarrow{k+1} (F\mathbf{x})_{n-k-1}))) \\
=& r_1 \cdot 0 = 0,
\end{aligned}
$$

from the definition of right multiplication in Section 6.2.4 (2). $\square$

(Case 2) Suppose $\bar{x}_{n-k} < n$. Then $\beta(\mathbf{x}, \bar{x}_{n-k}) = k + n - \bar{x}_{n-k}$ and $C^{k+1}_{n-k}(EF, \mathbf{x})$ is

$$PH((EF\mathbf{x})^{k+1}_{n-k})\{n - \bar{x}_{n-k}\}[n - \bar{x}_{n-k}] \oplus PH((EF\mathbf{x})^{k+1}_{n-k})\{n - \bar{x}_{n-k} + 1\}[n - \bar{x}_{n-k} + 1].$$

Note that $(EF\mathbf{x})^{q_{n-k}+1}_{n-k} = \mathbf{x}$ and there is a path from $\mathbf{x}$ to $(EF\mathbf{x})^{k+1}_{n-k}$ in $Q_{n,k}$:

$$\mathbf{x} = (EF\mathbf{x})^{q_{n-k}+1}_{n-k} \xrightarrow{q_{n-k}+1} (EF\mathbf{x})^{q_{n-k}+2}_{n-k} \xrightarrow{q_{n-k}+2} \cdots \xrightarrow{k} (EF\mathbf{x})^{k+1}_{n-k}.$$

Let $r_{I,EF}(\mathbf{x})$ be a product of the corresponding $n - \bar{x}_{n-k} = k - q_{n-k}$ generators in $H(R_{n,k})$. The right multiplication is a map $C(I, \mathbf{x}) \to C(EF, \mathbf{x})$ of left $H(R_n)$-modules defined on the generators by

$$m(I\mathbf{x}) \times (\rho(I, EF) \boxtimes e(\mathbf{x})) = r_{I,EF}(\mathbf{x}) \cdot m((EF\mathbf{x})^{k+1}_{n-k}).$$

A diagrammatic example is given in Figure 22.

Lemma 6.16. *The right multiplication by $\rho(I, EF) \boxtimes e(\mathbf{x})$ in Case (2) commutes with the differential.*

Proof. Since the differential on $C(I, \mathbf{x})$ is zero, it suffices to prove that

$$
\begin{aligned}
& d^{k+1}_{n-k}|_{hor}(EF, \mathbf{x})(m(I\mathbf{x}) \times (\rho(I, EF) \boxtimes e(\mathbf{x}))) \\
=& (r_{I,EF}(\mathbf{x}) \cdot m((EF\mathbf{x})^{k+1}_{n-k})) \times (e(E) \boxtimes r_F(\mathbf{x}; n-k)) \\
=& r_{I,EF}(\mathbf{x}) \cdot r_{EF}(\mathbf{x}; n-k, k+1) = 0,
\end{aligned}
$$

where $r_{EF}(\mathbf{x}; n-k, k+1)$ is a product of $q_{n-k} - q_{n-k-1} + 1$ generators in $H(R_{n,k})$ induced by the following path in $Q_{n,k}$:

$$(EF\mathbf{x})^{k+1}_{n-k} \xrightarrow{q_{n-k-1}+1} \cdots \xrightarrow{q_{n-k}+1} (EF\mathbf{x})^{k+1}_{n-k-1}.$$

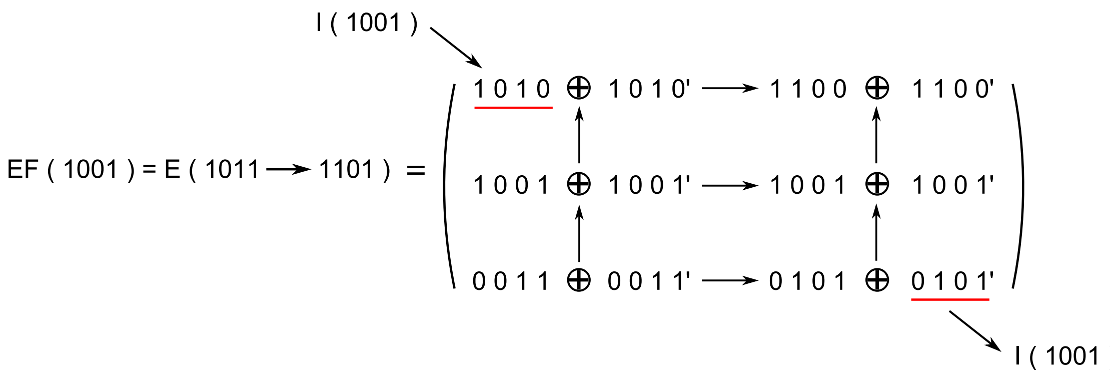


FIGURE 22. The map from $I(1010)$ to the top left corner is the right multiplication by $\rho(I, EF) \boxtimes e(\mathbf{x})$ when $\bar{x}_{n-k} < n$. The map from the bottom right corner to $I(1010)$ is the right multiplication by $\rho(EF, I) \boxtimes e(\mathbf{x})$ when $\bar{x}_1 > 1$.

Then $r_{I,EF}(\mathbf{x}) \cdot r_{EF}(\mathbf{x}; n-k, k+1)$ is induced by the concatenation of the two paths:

$$\mathbf{x} \xrightarrow{q_{n-k}+1} \cdots \xrightarrow{k} (EF\mathbf{x})^{k+1}_{n-k} \xrightarrow{q_{n-k-1}+1} \cdots \xrightarrow{q_{n-k}+1} (EF\mathbf{x})^{k+1}_{n-k-1}. \tag{15}$$

By using the commutation relation (R2) to rearrange the arrows, Equation (15) can be written as:

$$\mathbf{x} \xrightarrow{q_{n-k-1}+1} \cdots \xrightarrow{q_{n-k}} \mathbf{z}^0 \xrightarrow{q_{n-k}+1} \mathbf{z}^1 \xrightarrow{q_{n-k}+1} \mathbf{z}^2 \xrightarrow{q_{n-k}+2} \cdots \xrightarrow{k} (EF\mathbf{x})^{k+1}_{n-k-1}.$$

Hence,

$$r_{I,EF}(\mathbf{x}) \cdot r_{EF}(\mathbf{x}; n-k, k+1) = \cdots r(\mathbf{z}^0 \xrightarrow{q_{n-k}+1} \mathbf{z}^1) \cdot r(\mathbf{z}^1 \xrightarrow{q_{n-k}+1} \mathbf{z}^2) \cdots = 0$$

from the relation (R1). $\square$

6.4.3. *The right multiplication by* $\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x})$. Recall from Definition 5.17 that $\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x})$ exists if and only if $1 \leq i \leq k < n$ and $\mathbf{x} \in \mathcal{B}_{n,k}$ such that $x_i = n-k+i$. Note that the condition $x_i = n-k+i$ implies that $x_l = n-k+l$ for all $i \leq l \leq k$, i.e., each of the last $k-i+1$ states in $\mathbf{x}$ is $|1\rangle$. We are interested in $(i, n-k)$-th entry $C^i_{n-k}(EF, \mathbf{x})$ of the double complex (13):

$$PH((EF\mathbf{x})^i_{n-k})\{n-\bar{x}_{n-k}\}[n-\bar{x}_{n-k}+k-i+1] \oplus PH((EF\mathbf{x})^k_{n-k})\{n-\bar{x}_{n-k}+1\}[n-\bar{x}_{n-k}+k-i+2].$$

Note that $(EF\mathbf{x})^{q_{n-k}+1}_{n-k} = \mathbf{x}$ and $q_{n-k}+1 \leq i$. Hence, there is a path from $\mathbf{x}$ to $(EF\mathbf{x})^{k+1}_{n-k}$ through $(EF\mathbf{x})^i_{n-k}$ in $Q_{n,k}$:

$$\mathbf{x} = (EF\mathbf{x})^{q_{n-k}+1}_{n-k} \xrightarrow{q_{n-k}+1} \cdots \xrightarrow{i-1} (EF\mathbf{x})^i_{n-k} \xrightarrow{i} \cdots \xrightarrow{k} (EF\mathbf{x})^{k+1}_{n-k}.$$

Let $r_{I,EF}(\mathbf{x};i)$ be the product of $i-q_{n,k}-1$ generators in $H(R_{n,k})$ corresponding to the path from $\mathbf{x}$ to $(EF\mathbf{x})^i_{n-k}$. Then the right multiplication is a map $C(I,\mathbf{x}) \to C(EF,\mathbf{x})$ of left $H(R_n)$-modules defined on the generators by

$$m(I\mathbf{x}) \times (\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x})) = r_{I,EF}(\mathbf{x};i) \cdot m((EF\mathbf{x})^i_{n-k}),$$

where $m((EF\mathbf{x})^i_{n-k}) \in C^i_{n-k}(EF,\mathbf{x})$.

Example 6.17. Let $\mathbf{x} = (2,3) = |011\rangle \in \mathcal{B}_{3,2}$, then $x_i = n-k+i$ for $i = 1,2$ and $n = 3, k = 2$. Hence there exist

$$r_1 = \rho(I|011\rangle \xrightarrow{1} EF|011\rangle), \quad r_2 = \rho(I|011\rangle \xrightarrow{2} EF|011\rangle), \quad r_3 = \rho(I,EF) \boxtimes e(|011\rangle).$$

The right multiplications by r_1, r_2 and r_3 are described in Figure 23. More precisely,

$$m(I|011\rangle) \times r_3 = r(|011\rangle \xrightarrow{1} |101\rangle) \cdot r(|101\rangle \xrightarrow{2} |110\rangle) \cdot m(|110\rangle) \in C(EF,|011\rangle),$$
$$m(I|011\rangle) \times r_2 = r(|011\rangle \xrightarrow{1} |101\rangle) \cdot m(|101\rangle) \in C(EF,|011\rangle),$$
$$m(I|011\rangle) \times r_3 = m(|011\rangle) \in C(EF,|011\rangle).$$

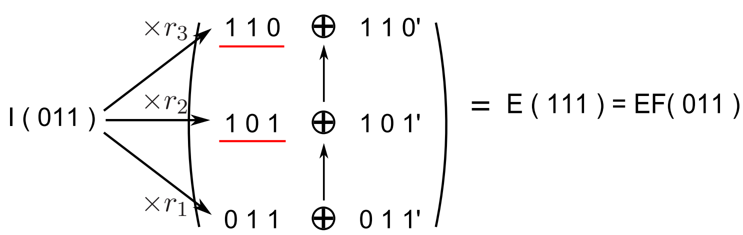


FIGURE 23.

We verify that the definition is compatible with the DG structure on $A \boxtimes R_n$.

Lemma 6.18. *For* $1 \le i \le k$ *and* $\mathbf{x} \in \mathcal{B}_{n,k}$ *with* $x_i = n-k+i$,

$$d(m(I\mathbf{x}) \times (\rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}))) = m(I\mathbf{x}) \times d(\rho(I\mathbf{x} \xrightarrow{k} EF\mathbf{x})).$$

Proof. For $i = k$, we give a diagrammatic proof for an explicit example: $r_1 = \rho(I|01\rangle \xrightarrow{1} EF|01\rangle)$ in Figure 24. Let $\rho = \rho(|01\rangle \xrightarrow{1} |01\rangle)$ and $r_0 = r(|01\rangle \xrightarrow{1} |10\rangle)$. Recall that $d(r_1) = r_2 \cdot r_3 + r_4 \cdot r_2$, here

$$r_2 = \rho(I,EF) \boxtimes e(|01\rangle), \quad r_3 = e(EF) \boxtimes \rho, \quad r_4 = e(I) \boxtimes \rho.$$

In Figure 24, the right multiplications by r_1, r_2 and r_3, r_4 are given in the left-hand and right-hand diagrams, respectively.

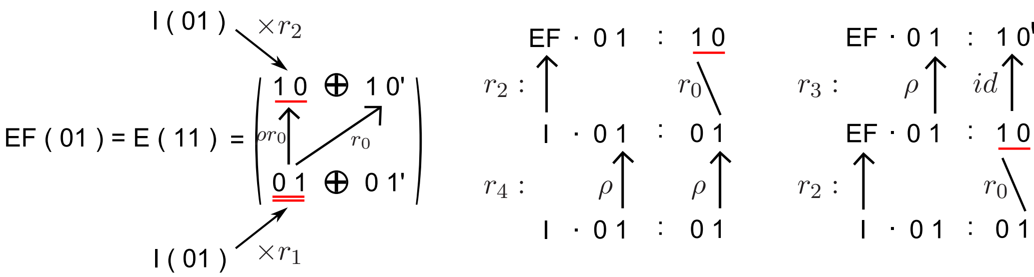


FIGURE 24. The map from $I(01)$ to the bottom left corner is the right multiplication by $\rho(I|01\rangle \xrightarrow{1} EF|01\rangle)$. The right-hand diagram gives the right multiplications by $d(\rho(I|01\rangle \xrightarrow{1} EF|01\rangle))$.

The right-hand side of the equation is

$$\begin{aligned} m(I|01\rangle) \times d(r_1) =& (m(I|01\rangle) \times r_4) \times r_2 + (m(I|01\rangle) \times r_2) \times r_3 \\ =& \rho \cdot r_0 \cdot m(|10\rangle) + r_0 \cdot m'(|10\rangle) \in C(EF, |01\rangle), \end{aligned}$$

which agrees with the left-hand side: $d(m(I|01\rangle) \times r_1) = d(m(|01\rangle)) \in C(EF, |01\rangle)$.

The proof for the case $i < k$ is similar. □

6.4.4. *The right multiplication by* $\rho(EF, I) \boxtimes e(\mathbf{x})$. The construction in this subsection is dual to that in Section 6.4.2. We discuss $(EF\mathbf{x})_1^1 \in \mathcal{B}_{n,k}$ shown in the bottom right corner $C_1^1(EF, \mathbf{x})$ of the double complex (13) depending on $\bar{x}_1 = 1$ or $\bar{x}_1 > 1$.

(Case 1) Suppose $\bar{x}_1 = 1$, i.e., the first state in the tensor product presentation is $|0\rangle$. Then we have $(EF\mathbf{x})_1^1 = \mathbf{x}$, $\beta(\mathbf{x}, \bar{x}_1) = 2k$ and $C_1^1(EF, \mathbf{x}) = PH(\mathbf{x})\{n-1\}[2k] \oplus PH(\mathbf{x})\{n\}[2k+1]$. The right multiplication

$$\times(\rho(EF, I) \boxtimes e(\mathbf{x})) : C(EF, \mathbf{x}) \to C(I, \mathbf{x})$$

is defined by the identity map from $PH(\mathbf{x})\{n\}[2k+1] \subset C_1^1(EF, \mathbf{x})$ to $PH(\mathbf{x}) = C(I, \mathbf{x})$. See Figure 21 for a diagrammatic example.

(Case 2) Suppose $\bar{x}_1 > 1$, then $\beta(\mathbf{x}, \bar{x}_1) = 2k - \bar{x}_1 + 1$ and $C_1^1(EF, \mathbf{x})$ is

$$PH((EF\mathbf{x})_1^1)\{n - \bar{x}_1\}[2k - \bar{x}_1 + 1] \oplus PH((EF\mathbf{x})_1^1)\{n - \bar{x}_1 + 1\}[2k - \bar{x}_1 + 2].$$

Note that $(EF\mathbf{x})_1^{q_1+1} = \mathbf{x}$ and there is a path from $(EF\mathbf{x})_1^1$ to $\mathbf{x}$ in $Q_{n,k}$:

$$(EF\mathbf{x})_1^1 \xrightarrow{1} (EF\mathbf{x})_1^2 \xrightarrow{2} \cdots \xrightarrow{q_1} \mathbf{x}.$$

Let $r_{EF,I}(\mathbf{x})$ be a product of the corresponding $q_1 = \bar{x}_1 - 1$ generators in $H(R_{n,k})$. The right multiplication is a map of left $H(R_n)$-modules: $C(EF, \mathbf{x}) \to C(I, \mathbf{x})$ defined on the generators by

$$m \times (\rho(EF, I) \boxtimes e(\mathbf{x})) = \begin{cases} r_{EF,I}(\mathbf{x}) \cdot m(I\mathbf{x}) & \text{if } m = m'((EF\mathbf{x})_1^1); \\ 0 & \text{otherwise}. \end{cases}$$

A diagrammatic example is given in Figure 22.

Lemma 6.19. *In both cases the right multiplication by $\rho(EF, I) \boxtimes e(\mathbf{x})$ commutes with the differential.*

Proof. The proof is similar to those of Lemmas 6.15 and 6.16. □

6.4.5. *The right multiplication by $\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})$.* The construction in this subsection is dual to that in Section 6.4.3. Recall from Definition 5.17 that $\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})$ exists if and only if $1 \leq j \leq k < n$ and $\mathbf{x} \in \mathcal{B}_{n,k}$ such that $x_j = j$. Note that the condition $x_j = j$ implies that $x_l = l$ for all $1 \leq l \leq j$, i.e., each of the first j states in $\mathbf{x}$ is $|1\rangle$. We are interested in $(j+1, 1)$-th entry $C_1^{j+1}(EF, \mathbf{x})$ of the double complex (13):

$$PH((EF\mathbf{x})_1^{j+1})\{n - \bar{x}_1\}[2k - \bar{x}_1 + j - 1] \oplus PH((EF\mathbf{x})_1^{j+1})\{n - \bar{x}_1 + 1\}[2k - \bar{x}_1 + j].$$

Note that $(EF\mathbf{x})_1^{q_1+1} = \mathbf{x}$. Hence, there is a path from $(EF\mathbf{x})_1^1$ to $\mathbf{x}$ through $(EF\mathbf{x})_1^{j+1}$ in $Q_{n,k}$:

$$(EF\mathbf{x})_1^1 \xrightarrow{1} \cdots \xrightarrow{j} (EF\mathbf{x})_1^{j+1} \xrightarrow{j+1} \cdots \xrightarrow{q_1} (EF\mathbf{x})_1^{q_1+1} = \mathbf{x}.$$

Let $r_{EF,I}(\mathbf{x}; j+1)$ be a product of $q_1 - j$ generators in $H(R_{n,k})$ corresponding to the path from $(EF\mathbf{x})_1^{j+1}$ to $\mathbf{x}$. Then the right multiplication is a map $C(EF, \mathbf{x}) \to C(I, \mathbf{x})$ of left $H(R_n)$-modules defined on the generators by

$$m \times (\rho(EF, I) \boxtimes e(\mathbf{x})) = \begin{cases} r_{EF,I}(\mathbf{x}; j+1) \cdot m(I\mathbf{x}) & \text{if } m = m'((EF\mathbf{x})_1^{j+1}); \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 6.20. *The right multiplication is compatible with the DG structure on $A \boxtimes R_n$:*

$$d(m'((EF\mathbf{x})_1^{j+1})) \times (\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})) = m'((EF\mathbf{x})_1^{j+1}) \times d((\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x}))),$$

for $1 \leq j \leq k < n$ and $\mathbf{x} \in \mathcal{B}_{n,k}$ such that $x_j = j$.

Proof. The proof is similar to that of Lemma 6.18. □

This concludes the definition of the right $A \boxtimes R_n$-module structure on C.

Proposition 6.21. *The definitions of the right multiplications by $A \boxtimes R_n$ is well-defined.*

Proof. (1) For the relatioons in $A \boxtimes R_n$, we need to verify that

$$(m \times r_1) \times r_2 = (m \times r_1') \times r_2',$$

if $r_1 \cdot r_2 = r_1' \cdot r_2' \in A \boxtimes R_n$ for $m \in C$ and $r_1, r_2, r_1', r_2' \in A \boxtimes R_n$. We checked the relations $(e(\Gamma) \boxtimes \rho(\mathbf{x} \xrightarrow{i} \mathbf{x}))^2 = 0$ for $\Gamma = E, F$ in Lemmas 6.5 and 6.9. The commutation relations which come from isotopies of stackings of disjoint rook diagrams are easily verified since the definition of the right multiplication only depends on local properties of rook diagrams.

(2) For the DG structure, we need to verify that

$$d(m \times r) = d(m) \times r + m \times d(r),$$

for $m \in C$ and any generator $r \in A \boxtimes R_n$. We proved it when

- $d(r) = 0$ in Lemmas 6.6, 6.7, 6.10, 6.11, 6.15, 6.16, and 6.19;
- $r = e(F) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ with $s_1 = 0$ in Lemma 6.8;
- $r = e(E) \boxtimes r(\mathbf{x} \xrightarrow{i,s_1,\mathbf{v}} \mathbf{y})$ with $s_0(\mathbf{v}) = 0$ in Lemma 6.12;
- $r = \rho(I\mathbf{x} \xrightarrow{i} EF\mathbf{x}))$ and $\rho(EF\mathbf{x} \xrightarrow{j} I\mathbf{x})$ in Lemmas 6.18 and 6.20, respectively.

The proof for other cases is similar and we leave it to the reader. □

It is easy to see that the left $H(R_n)$-module structure and the right $A \boxtimes R_n$-module structure on C_n are compatible:

$$a \cdot (m \times r) = (a \cdot m) \times r,$$

for $a \in H(R_n), r \in A \boxtimes R_n$ and $m \in C_n$. Hence C_n is a t-graded DG $(H(R_n), A \boxtimes R_n)$-bimodule.

7. The categorical action of $DGP(A)$ on $DGP(R_n)$

In this section, we use the bimodule C_n defined above to categorify the action of $\mathbf{U}_T$ on V_n.

Definition 7.1. Let $\eta_n : DGP(A \boxtimes R_n) \xrightarrow{C_n \otimes_{A \boxtimes R_n} -} DGP(H(R_n))$ be a functor of tensoring with the DG $(H(R_n), A \boxtimes R_n)$-bimodule C_n over $A \boxtimes R_n$.

Lemma 7.2. *The functor η_n maps $P(\Gamma, \mathbf{x})$ to $C_n(\Gamma, \mathbf{x}) \in DGP(H(R_n))$, for all $\Gamma \in \mathcal{B}$ and $\mathbf{x} \in \mathcal{B}_n$.*

Proof. The proof is similar to that of Lemma 2.21. □

There is an induced exact functor η_n between the 0-th homology categories:

$$\eta_n : H^0(DGP(A \boxtimes R_n)) \xrightarrow{C_n \otimes_{A \boxtimes R_n} -} H^0(DGP(H(R_n))).$$

We choose some equivalences between triangulated categories:

$$\mathcal{G}_n : H^0(DGP(H(R_n))) \to H^0(DGP(R_n)),$$
$$\mathcal{F}_n : H^0(DGP(A \otimes R_n)) \to H^0(DGP(A \boxtimes R_n)),$$

which induce isomorphisms on the Grothendieck groups. Let $\mathcal{M}_n$ be the composition:

$$\mathcal{M}_n = \mathcal{G}_n \circ \eta_n \circ \mathcal{F}_n \circ \chi_n : H^0(DGP(A)) \times H^0(DGP(R_n)) \to H^0(DGP(R_n)),$$

where

$$\chi_n : H^0(DGP(A)) \times H^0(DGP(R_n)) \to H^0(DGP(A \otimes R_n))$$

induces the tensor product on the Grothendieck groups.

Proof of Theorem 1.3. We use $\{[P(\Gamma, \mathbf{x})]\}$ as a basis of $K_0(H^0(DGP(A \boxtimes R_n)))$ to compute $K_0(\eta_n)$. By Lemma 6.14,

$$K_0(\eta_n)(\Gamma \otimes \mathbf{x}) = K_0(\eta_n)([P(\Gamma, \mathbf{x})]) = [C \otimes P(\Gamma, \mathbf{x})] = [C(\Gamma, \mathbf{x})] = \Gamma(\mathbf{x}) \in V_n,$$

Hence, the $\mathbb{Z}[t^{\pm 1}]$-linear map

$$K_0(\mathcal{M}_n) : K_0(H^0(DGP(A))) \times K_0(H^0(DGP(R_n))) \to K_0(H^0(DGP(R_n)))$$

agrees with the action of $\mathbf{U}_T$ on V_n: $\mathbf{U}_T \times V_n \to V_n$. □

Remark 7.3. It is natural to ask whether the categorical action is associative up to equivalence:

$$\begin{array}{ccc} H^0(DGP(A)) \times H^0(DGP(A)) \times H^0(DGP(R_n)) & \xrightarrow{id \times \mathcal{M}_n} & H^0(DGP(A)) \times H^0(DGP(R_n)) \\ \downarrow{\scriptstyle \mathcal{M} \times id} & & \downarrow{\scriptstyle \mathcal{M}_n} \\ H^0(DGP(A)) \times H^0(DGP(R_n)) & \xrightarrow{\mathcal{M}_n} & H^0(DGP(R_n)). \end{array}$$

The question is equivalent to verifying some associativity relation on various DG bimodules. The computation is quite technical and we leave it to future work.

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UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089

E-mail address: yintian@usc.edu